Badji Mokhtar University -Annaba
Faculty of Sciences
Physics Department

Mechanics of the material point:

Courses and Corrected Exercises

Intended for first-year students in the Bachelor's degree programs of Science and Technology (ST) and Matter Science (MS).

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Preamble

This handout has been specifically designed for first-year students in the common core of Sciences and Technologies (ST) and Material Sciences (MS). Its purpose is to provide essential foundations in dimensional analysis, vector calculus, point kinematics, and dynamics of a material point in a Galilean reference frame. These concepts are fundamental for studies in the fields of science and technology.

The handout we present to you covers two essential topics in physics: dimensional analysis and vector calculus, as well as point kinematics and dynamics of a material point in a Galilean reference frame.

The first part of the handout focuses on dimensional analysis, a fundamental method for studying the relationships between physical quantities. You will discover the basics of dimensional analysis, including its definition and use, as well as the dimensions of the International System of Units. This section will allow you to acquire key skills for formulating and understanding physical laws.

Next, you will delve into vector calculus, which is essential for describing physical quantities that have both direction and magnitude. You will learn about scalar and vector quantities, as well as the different graphical representations of vectors. You will also explore fundamental operations on vectors, such as dot product and cross product, and discover their remarkable properties.

The second part of the handout is dedicated to point kinematics, which studies the motion of objects without considering the forces causing them. You will be introduced to the basic concepts of kinematics, such as the positioning of a material point in space and time, coordinate systems, velocity and acceleration vectors, as well as the Frenet frame to describe curved motions.

Moreover, you will explore kinematics with a change of reference frame, where you will study the relative and absolute motions between different frames of reference. You will learn how to describe motions using derivations in a moving frame and understand methods for composing velocities and accelerations.

Finally, the third part of the handout addresses the dynamics of a material point in a Galilean reference frame. You will examine the fundamental principles of dynamics, including the principle of inertia, the action-reaction principle, and the fundamental relation of dynamics. You will also understand the concept of angular momentum and its important properties. Additionally, you will explore different forms of energy (kinetic, potential, and mechanical) and their relationship to work and conservative forces.

The handout is accompanied by a series of exercises to allow you to practice the presented concepts and methods. Detailed solutions are also provided to assist you in your learning.

We hope that this handout will provide you with a solid understanding of dimensional analysis, vector calculus, point kinematics, and dynamics of a material point in a Galilean reference frame. This knowledge is essential for studying physics and comprehending the natural phenomena that surround us. Enjoy reading and happy learning!

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Chapiter I

Dimensional Analysis and Vector Calculus

I.1. Dimensional Analysis

1.1 Introduction

When providing a measurement or calculation result, it is imperative to indicate the nature of the obtained result. This is known as the dimension of the quantity considered. This dimension can be referenced by a recognized standard, which assigns it a unit. If we measure the distance between two cities, the dimension of the result is length. This can be expressed numerically in various different units: meters, feet, inches, fathoms, leagues, etc. Regardless of the chosen unit of representation, the distance will remain the same, and its dimension will always be length. By indicating the unit, we specify both the dimension and the reference system chosen as the unit. Therefore, it is essential to always provide a unit for any numerical result.

Example: If we measure the distance L between two cities, we will present the numerical result as follows:

$$L = 35\ Km$$

The number **35**, on its own, would have absolutely no meaning. It is precisely because this numerical result is given with a unit that it has meaning. This unit serves to provide both the reference of the measurement (in this case, kilometers) and to specify the nature of the result - a distance. To clarify, let's say for now that a numerical result will necessarily involve the use of a unit, while a literal result will necessarily involve dimensions. This chapter consists of two parts. The first part is dedicated to dimensional analysis, and the second part is dedicated to vector calculus.

1.2 Dimensional Analysis

1.2.1 Definition

Dimension: In experimental physics, every measured quantity must be described primarily in a qualitative manner: length, time, mass, etc. This qualitative aspect is the dimension of the measured quantity.

Dimensional Analysis: A dimension can be expressed in terms of other dimensions by simply applying its definition. For example, the dimensions $[\boldsymbol{V}]$ and $[\boldsymbol{a}]$ of the velocity and acceleration of a moving object, represented by the position vector $\overrightarrow{\boldsymbol{OM}}$, can be obtained in terms of the dimensions of time and length by applying their respective definitions:

$$\vec{V} = \frac{d\overrightarrow{OM}}{dt} \quad \text{Let's consider } [V] = \frac{[L]}{[T]} \quad \text{and } \vec{a} = \frac{d\vec{V}}{dt} \; Let's\ consider \quad [V] = \frac{[V]}{[T]} = \frac{[L]}{[T]^2}$$

Dimensional analysis involves establishing relationships between different dimensions through physical laws or known relationships.

1.2.2 Dimensions of the International System of Units:

In the field of Physical Sciences, there are seven fundamental and independent dimensions from which all others are defined. These dimensions are:

1. Length (L)
2. Mass (M)
3. Time (T)
4. Electric Current (I)
5. Temperature (Θ)
6. Amount of Substance (N)
7. Luminous Intensity (J)

These seven dimensions form the basis of the International System of Units (SI) and provide a framework for measuring and quantifying various physical quantities.

Physical quantity	**SI Unit**	**Symbol**	**Dimension**
Time	second	s	$[T]$
Mass	Kilogram	kg	$[M]$
Length	meter	m	$[L]$
Electric Current	ampere	A	$[I]$
Temperature	kelvin	K	$[Q]$ ou $[\Theta]$
Amount of Substance	mole	mol	$[n]$
Luminous Intensity	candela	cd	$[I]_L$

The table below provides some derived quantities (which are derived from fundamental units) along with their units.

1.2.3 Using Dimensional Analysis:

- On either side of a literal equality, the quantities on the left and right of the equal sign ***must necessarily*** have the same dimension.
- It ***is not possible*** to add or subtract quantities with different dimensions.
- The quotient (respectively product) of two quantities with different dimensions has a dimension that is the quotient (respectively product) of the dimensions. The ratio of two quantities with the same dimension is therefore dimensionless. It is said to be ***adimensional***.
- The argument of common mathematical functions (exp, cos, sin, etc.) has no dimension. In the physical sciences, in particular, when using these functions, it is important to ensure that the argument being used is dimensionless.
- The dimension of a derivative is the quotient of the dimensions of the derived function and the derivative's argument.

These guidelines highlight the importance of maintaining consistency in dimensions when performing mathematical operations and using functions in the physical sciences. By applying dimensional analysis, we can ensure the meaningfulness and accuracy of calculations and equations.

Physical quantity	**Dimensional Equation**	**Basic unit**	**Names**
Force	MLT^{-2}	$kg.\ m.\ s^{-2}$	newton : N
Pressure	$ML^{-1}T^{-2}$	$kg.\ m^{-1}.\ s^{-2}$	pascal : Pa
Work	$ML^{2}T^{-2}$	$kg.\ m^{2}.\ s^{-2}$	joule : J
Power	$ML^{2}T^{-3}$	$kg.\ m^{2}.\ s^{-3}$	watt : W
Charge	Q = IT	A. s	coulomb : C
Potential	$ML^{2}T^{-2}Q^{-1}$	$kg.\ m^{2}.\ s^{-3}\ A^{-1}$	volt : V
Capacitance	$M^{-1}L^{-2}T^{2}Q^{2}$	$kg^{-1}\ m^{-2}\ s^{4}\ A^{2}$	farad : F
Resistance	$ML^{2}T^{-1}Q^{-2}$	$kg.\ m^{2}.\ s^{-3}\ A^{-2}$	ohm : Ω
Conductance	$M^{-1}L^{-2}TQ^{2}$	$kg^{-1}.\ m^{-2}.\ s^{3}\ A^{2}$	siemens : S
Magnetic field	$MT^{-1}Q^{-1}$	$kg\ s^{-2}\ A^{-1}$	tesla : T
Inductance	$ML^{2}T^{-2}I^{-2}$	$kg\ m^{2}\ s^{-2}\ A^{-2}$	henry : H

These equations represent the dimensions of various physical quantities along with their corresponding basic units and names. It is important to understand these dimensional equations and units in order to accurately represent and analyze physical phenomena in different fields of science and engineering.

Important Note: It is crucial not to confuse units and dimensions. When we provide a unit, we implicitly specify a dimension. However, the converse is not true. Taking the example from the introduction of this chapter, the distance L separating the two cities can be written as:

L = 35 km = 35$\times$103 m = 35$\times$105 cm = 35$\times$106 mm = 35$\times$109μ m = 21,9 mi (US mile)

In these different equalities, the only change is a change in unit. This change results in a modification of the numerical value presented **but not the nature of the measured quantity** L, which remains a **distance**.

In Summary: Dimensional analysis allows us to:

- Verify the consistency of theoretical calculations by ensuring that the derived relationship is dimensionally possible (the quantities on both sides of the equality sign must have the same dimension).
- Express the dimension of relevant numbers in physics (especially fundamental constants) and thus determine their unit within the chosen system.

By employing dimensional analysis, we can check the validity of equations, establish the units of physical quantities, and ensure the accuracy and coherence of calculations in various scientific and engineering disciplines.

Vector Calculus

2.1 Scalar Quantity

A scalar quantity is always expressed by a numerical value followed by the corresponding unit.

Example: volume, mass, temperature, electric charge, energy…

Scalar quantities represent physical quantities that have magnitude but do not have a specific direction. They can be fully described by their numerical value and unit, without any additional vector or directional information. Scalar quantities are fundamental in many areas of physics and provide essential information for various calculations and analyses.

2.2 Grandeur vectorielle

On appelle grandeur vectorielle toute grandeur qui nécessite un sens, une direction, un point d'application en plus de sa valeur numérique appelée intensité ou module.

Exemple **:** le déplacement, la vitesse, la force, le champ électrique…

2.2 Vector Quantity

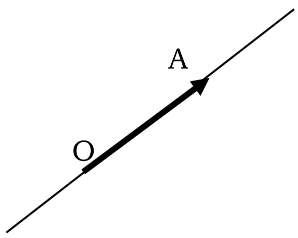
A vector quantity is any quantity that requires a sense, a direction, and an application point in addition to its numerical value, known as magnitude or modulus. Example: displacement, velocity, force, electric field…

Vector quantities possess both magnitude and direction, and they are represented by vectors. In addition to the numerical value, vector quantities provide information about the specific direction in which the quantity is acting or being measured. This directional information is crucial for accurately representing and analyzing physical phenomena involving vector quantities, such as motion, forces, and fields.

2.3 Graphical Representation of a Vector:

A vector is represented graphically by a line segment OA, where O is the chosen origin and A is the endpoint. A vector is defined by:

- Its origin (starting point).
- Its direction.
- Its sense (arrowhead indicating the direction).
- Its magnitude (length of the line segment).



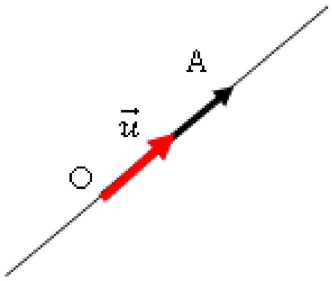
The graphical representation of a vector visually conveys its essential characteristics, allowing us to understand its starting point, direction, sense, and magnitude. This representation is commonly used to depict vectors in various fields of science and engineering, aiding in the visualization and analysis of vector quantities.

$\overrightarrow{OA}$ represents the vector, including its four characteristics.

$\|\overrightarrow{OA}\| = |\overrightarrow{OA}| = OA$ represents the magnitude or intensity of the vector.

The notation $\overrightarrow{OA}$ indicates the vector with its origin and endpoint, while $\|\overrightarrow{OA}\|$ or $|\overrightarrow{OA}|$ or OA represents the magnitude or length of the vector. The magnitude is a scalar value that represents the numerical value or intensity of the vector, disregarding its direction. It provides information about the size or magnitude of the vector without considering its orientation or position.

2.4 Unit Vector:



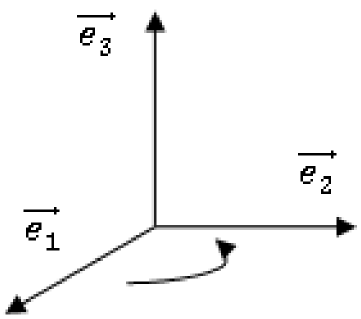
An unit vector is a vector whose magnitude is equal to one (1). It is often denoted by placing a caret (^) or a hat symbol (∧) on top of the vector symbol, such as $\vec{u}$ or $\hat{u}$.

A unit vector represents the direction of a vector without any consideration for its magnitude or intensity. It is used to indicate the orientation or directionality of a vector. By multiplying a vector by its corresponding unit vector, we can express a vector parallel to the unit vector. For example, $\overrightarrow{OA} = OA\vec{u}$ represents a vector parallel to the unit vector $\vec{u}$ ⃗, with a magnitude given by OA.

2.5 Vector Components:

Let's consider a basis in the space R^3 denoted as $R_o = (O, \vec{e_1}, \vec{e_2}, \vec{e_3})$.

This basis is orthonormal if:



- The vectors $\vec{e_1}$, $\vec{e_2}$, and $\vec{e_3}$ are mutually orthogonal, meaning that they are perpendicular to each other.
- Each vector has a magnitude of 1, making them unit vectors.

$$\vec{e_i}.\vec{e_j} = \begin{cases} 1 & si \quad i = j \\ 0 & si \quad i \neq j \end{cases}$$

In an orthonormal basis, any vector $\vec{v}$ in R^3 can be expressed as a linear combination of the basis vectors $\overrightarrow{e_1}$, $\overrightarrow{e_2}$, and $\overrightarrow{e_3}$. The coefficients of this linear combination are called the components of the vector. They represent how much of each basis vector contributes to the overall vector.

The components of a vector can be determined using scalar projections or dot products. By decomposing a vector into its components, we can analyze and manipulate the vector more easily in various calculations and applications.

In this basis, a vector $\vec{v}$ with components $(x, y, z) \in R^3$ can be written as:

$$\vec{v} = x\overrightarrow{e_1} + y\overrightarrow{e_2} + z\overrightarrow{e_3}$$

The real quantities $x, y,$ and z are called the components of the vector $\vec{v}$ in the basis R^3. The notation commonly used to represent the vector with its components in the R^3 basis is:

$$\vec{v} = \underset{R_0}{} \begin{cases} x \\ y \\ z \end{cases}$$

This notation indicates the vector $\vec{v}$ with a square bracket enclosing the components (x, y, z) arranged in the same order as the basis vectors $\overrightarrow{e_1}$, $\overrightarrow{e_2}$, and $\overrightarrow{e_3}$.

2.6 Internal Composition Law: Vector Addition

The sum of two vectors $\vec{v}_1$ and $\vec{v}_2$ is a vector $\vec{w}$ such that:

$\forall\ \vec{v}_1, \vec{v}_2$, we have $\vec{w} = \vec{v}_1 + \vec{v}_2 \in R^3$

Let (a_1, a_2, a_3) be the components of the vector $\vec{v}_1$, so $\vec{v}_1 = a_1\overrightarrow{e_1} + a_2\overrightarrow{e_2} + a_3\overrightarrow{e_3}$, and let (b_1, b_2, b_3) be the components of the vector $\vec{v}_2$, so $\vec{v}_2 = b_1\overrightarrow{e_1} + b_2\overrightarrow{e_2} + b_3\overrightarrow{e_3}$.

The vector sum is defined by the relation:

$$\vec{w} = \vec{v}_1 + \vec{v}_2 = (a_1 + b_1)\overrightarrow{e_1} + (a_2 + b_2)\overrightarrow{e_2} + (a_3 + b_3)\overrightarrow{e_3}$$

The neutral element or zero vector is denoted as $\vec{0} = (0, 0, 0)$. It has zero components in all directions.

2.6.1 Properties of Vector Addition

- Vector addition is commutative: $\vec{v}_1 + \vec{v}_2 = \vec{v}_2 + \vec{v}_1$. The order in which vectors are added does not affect the result.
- Vector addition is associative: $(\vec{v}_1 + \vec{v}_2) + \vec{v}_3 = \vec{v}_1 + (\vec{v}_2 + \vec{v}_3)$. The grouping of vectors does not affect the result.

- The zero vector acts as the neutral element: $\vec{v} + \vec{0} = \vec{v}$. Adding the zero vector to any vector does not change the vector.
- For every vector $\vec{v}$, there exists an opposite vector denoted as $-\vec{v}$, such that $\vec{v} + (-\vec{v}) = \vec{0}$. Adding a vector to its opposite results in the zero vector

2.6.2 Scalar Multiplication

If λ is a real number and $\vec{v}$ is a vector, their product is a vector.

$\forall\ \lambda\ \in R, \forall\ \vec{v} \in R^{3} \Rightarrow \vec{w} = \lambda\vec{v}\ \in\ R^{3}$

The vector $\vec{w}$ is collinear with the vector $\vec{v}$.

If the vector $\vec{v}$ has components (a, b, c) such that $\vec{v} = a_1\vec{e_1} + a_2\,\vec{e_2} + a_3\vec{e_3}$, the vector $\vec{w}$ can be written as:

$$\vec{w} = \lambda\, a_1\vec{e_1} + \lambda\, a_2\,\vec{e_2} + \lambda\, a_3\vec{e_3}$$

Multiplying a vector by a scalar satisfies the following properties:

- *Distributivity with respect to scalar addition:* $(\lambda_1 +: \lambda_2)\vec{v} = \lambda_1\vec{v} + \lambda_2\vec{v}$
- *Distributivity with respect to vector addition:* $\lambda(\vec{v_1} + \vec{v_2}) = \lambda\vec{v_1} + \lambda\vec{v_2}$
- *Associativity for scalar multiplication:* $\lambda_1(\lambda_2\vec{v}) = \lambda_1\lambda_2\vec{v}$

2.7 Dot Product of Two Vectors (scalar product): The dot product of two vectors $\vec{v}_1$ and $\vec{v}_2$ is an external composition law that associates with the two vectors a scalar (real number), denoted as $\vec{v}_1.\vec{v}_2$, such that:

$$\forall\ \vec{v}_1, \vec{v}_2 \in R^{3} \Rightarrow \vec{v}_1.\vec{v}_2 \in\ R$$

$$\vec{v}_1 \cdot \vec{v}_2 = \|\vec{v_1}\|\|\vec{v_2}\|\ \cos(\vec{v}_1, \vec{v}_2)$$

Here, $\|\vec{v_1}\|$ and $\|\vec{v_2}\|$ represent the magnitudes (or lengths) of the vectors $\vec{v}_1$ and $\vec{v}_2$, respectively, and cos(angle between $\vec{v}_1$ and $\vec{v}_2$ is the cosine of the angle formed between the two vectors.

The result of a dot product is a scalar. The dot product is zero if:

- The two vectors are orthogonal, meaning they are perpendicular to each other: $(\vec{v}_1\ \perp \vec{v}_2)$.
- One of the vectors is the zero vector, which has no direction or magnitude.

2.7.1 properties of dot product

The dot product satisfies the following properties:

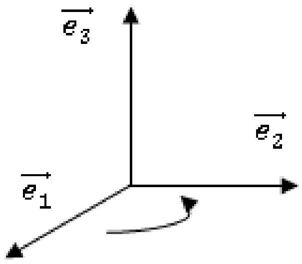
- *Linearity:* $(\vec{v}_1 + \vec{v}_2).\vec{w} = \vec{v}_1.\vec{w} + \vec{v}_2.\vec{w}$.
- *Symmetry with respect to the vectors:* $\vec{v}.\vec{w} = \vec{w}.\vec{v}$. *Therefore,* $\vec{v}.\vec{v} > 0$ *if* $\vec{v} \neq \vec{0}$. *The dot product is a symmetric bilinear form associated with the vectors* $\vec{v}$ *and* $\vec{w}$.

2.7.2 Analytical expression of the dot product

A base of the space R^3, denoted as $R_o = (O, \vec{e_1}, \vec{e_2}, \vec{e_3})$, is orthonormal if the vectors in the base are mutually orthogonal and have a norm equal to 1. This means that:

$$\vec{e_i}.\vec{e_j} = \begin{cases} 1 & si \quad i = j \\ 0 & si \quad i \neq j \end{cases}$$

The basis is said to be right-handed if an observer standing at the end of the vector $\vec{e_3}$ will see the vector turning towards the vector $\vec{e_1}$ in the counterclockwise direction.



Let's consider two vectors $\vec{v}_1$ and $\vec{v}_2$.. Their expressions in this basis are:

$$\vec{v}_1 = a_1\vec{e_1} + a_2\vec{e_2} + a_3\vec{e_3}\,.$$
$$\vec{v}_2 = b_1\vec{e_1} + b_2\vec{e_2} + b_3\vec{e_3}\,.$$

The dot product of the two vectors is given by:

$$\vec{v}_1.\vec{v}_2 = (a_1\vec{e_1} + a_2\vec{e_2} + a_3\vec{e_3}).(b_1\vec{e_1} + b_2\vec{e_2} + b_3\vec{e_3}) = a_1b_1 + a_2b_2 + a_3b_3$$

In other words, the dot product is equal to the sum of the products of the corresponding components of the vectors.

2.7.3 The norm or magnitude of a vector

$\vec{v}$ is denoted as $||\vec{v}||$ and is defined as the positive square root of the dot product of the vector with itself: $||\vec{v}|| = \sqrt{\vec{v}.\vec{v}} = \sqrt{\vec{v}^2}$

We have the following properties of the vector norm:

- $||\lambda\vec{v}|| = ||\lambda||||\vec{v}||$, where λ is a scalar.

- $\|\vec{v}_1\| - \|\vec{v}_2\| \leq \|\vec{v}_1 + \vec{v}_2\| \leq \|\vec{v}_1\| + \|\vec{v}_2\|$, which is known as the triangle inequality.

These properties allow us to measure the length or magnitude of a vector and compare the sizes of different vectors.

2.7.4. Orthogonal vectors

Two vectors are said to be orthogonal if and only if their dot product is zero:

If: $\vec{v} \perp \vec{w} \Leftrightarrow \vec{v}\,.\vec{w} = \vec{0}$

If three non-zero vectors are pairwise orthogonal, they are linearly independent and form an orthogonal basis in R^3.

2.7.5. Orthonormal basis

A basis is said to be orthonormal if its constituent vectors are mutually perpendicular and have unit norms. If $b = (\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3})$ is an orthonormal basis, then we have:

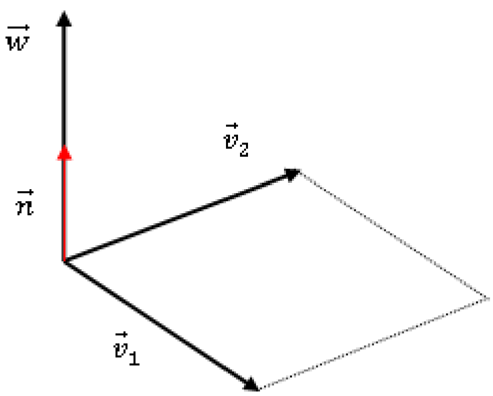
$\overrightarrow{e_1}.\overrightarrow{e_2} = 0,\qquad \overrightarrow{e_1}.\overrightarrow{e_3} = 0,\qquad \overrightarrow{e_2}.\overrightarrow{e_3} = 0$

$\overrightarrow{e_1}.\overrightarrow{e_1} = \overrightarrow{e_1}^2 = 1,\qquad \overrightarrow{e_2}.\overrightarrow{e_2} = \overrightarrow{e_2}^2 = 1,\qquad \overrightarrow{e_3}.\overrightarrow{e_3} = \overrightarrow{e_3}^2 = 1$

2.8 Cross product of two vectors

The cross product of two vectors $\vec{v}_1$ and $\vec{v}_2$ in three-dimensional space R^3 is a vector $\vec{w}$ defined as:

$$\vec{w} = \vec{v}_1 \wedge \vec{v}_2 = \|\vec{v}_1\|\|\vec{v}_2\| \sin(\vec{v}_1, \vec{v}_2)\,\vec{n}$$



The vector $\vec{w}$ is perpendicular to both $\vec{v}_1$ and $\vec{v}_2$ and is determined by the right-hand rule (or the curl rule). The magnitude of the vector $\vec{w}$ is given by the area of the parallelogram formed by $\vec{v}_1$ and $\vec{v}_2$.

$\vec{n}$ is a unit vector perpendicular to both $\vec{v}_1$ and $\vec{v}_2$. The cross product is zero if:

- *The two vectors are collinear (parallel or antiparallel).*
- *One of the vectors is zero (the zero vector).*

2.8.1 Properties of the Cross Product:

- The magnitude of the cross product is equal to the area of the parallelogram formed by $\vec{v}_1$ and $\vec{v}_2$.

  The cross product is distributive over vector addition:

$$(\vec{v}_1 + \vec{v}_2) \wedge \vec{w} = \vec{v}_1 \wedge \vec{w} + \vec{v}_2 \wedge \vec{w}$$
$$\vec{w} \wedge (\vec{v}_1 + \vec{v}_2) = \vec{w} \wedge \vec{v}_1 + \vec{w} \wedge \vec{v}_2$$

  The cross product is associative with scalar multiplication:

$$(\lambda\, \vec{v}) \wedge \vec{w} = \lambda(\vec{v} \wedge \vec{w})$$
$$\vec{v} \wedge (\lambda \vec{w}) = \lambda(\vec{v} \wedge \vec{w})$$

- The cross product is antisymmetric (anticommutative):

$$\vec{v}_1 \wedge \vec{v}_2 = -\, \vec{v}_2 \wedge \vec{v}_1$$

- Applying this property to the cross product of the same vector:

$$\vec{v} \wedge \vec{v} = -\,(\vec{v} \wedge \vec{v}) = \vec{0}$$

From this property, it can be deduced that two non-zero vectors are collinear if and only if their cross product is zero.

If $\quad \vec{v}_1 \parallel \vec{v}_2 \quad$ then $\quad \vec{v}_1 \wedge \vec{v}_2 = \vec{0}$.

Indeed, if v ⃗_1∥v ⃗_2, we can write:

$$\vec{v}_1 = \lambda \vec{v}_2 \quad \Longrightarrow \quad \vec{v}_1 \wedge \vec{v}_2 = \lambda(\vec{v}_2 \wedge \vec{v}_2) = \vec{0}$$

2.8.2 Cross product of unit vectors in an orthonormal basis

If $b = (\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3})$ is an orthonormal basis, then we have:

Direct orientation: $\overrightarrow{e_1} \wedge \overrightarrow{e_2} = \overrightarrow{e_3}$, $\quad \overrightarrow{e_2} \wedge \overrightarrow{e_3} = \overrightarrow{e_1}$, $\quad \overrightarrow{e_3} \wedge. \overrightarrow{e_1} = \overrightarrow{e_2}$

Opposite orientation: $\overrightarrow{e_2} \wedge \overrightarrow{e_1} = -\overrightarrow{e_3}$, $\quad \overrightarrow{e_3} \wedge \overrightarrow{e_2} = -\overrightarrow{e_1}$, $\quad \overrightarrow{e_1} \wedge. \overrightarrow{e_3} = -\overrightarrow{e_2}$

2.8.3 Analytical expression of the cross product in a direct orthonormal basis R

The cross product of two vectors with respective components in a direct orthonormal basis R is given by:

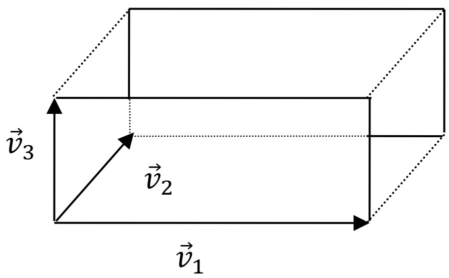
$$\vec{v}_1 = \underset{R}{\begin{cases} x_1 \\ y_1 \\ z_1 \end{cases}} \quad \text{et} \quad \vec{v}_2 = \underset{R}{\begin{cases} x_2 \\ y_2 \\ z_2 \end{cases}}$$

$$\vec{v}_1 \wedge \vec{v}_2 = \begin{cases} x_1 \\ y_1 \\ z_1 \end{cases} \wedge \; . \begin{cases} x_2 \\ y_2 \\ z_2 \end{cases} = \begin{cases} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{cases}$$

2.8.4 Scalar Triple Product

The scalar triple product of three vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ taken in that order is defined as the real number given by:

$$\vec{v}_1 . (\vec{v}_2 \wedge \vec{v}_3)$$



The scalar triple product is a scalar quantity equal to the volume of the parallelepiped formed by the three vectors.

The scalar triple product is zero if:

- *The three vectors lie in the same plane.*
- *Two of the vectors are collinear.*
- *One of the vectors is zero.*

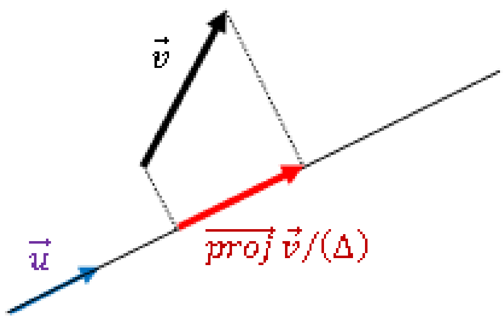
It can be easily shown that in a direct orthonormal basis, the scalar triple product is invariant under cyclic permutation of the three vectors, since the dot product is commutative:

$$\vec{v}_1 . (\vec{v}_2 \wedge \vec{v}_3) = \vec{v}_3 . (\vec{v}_1 \wedge \vec{v}_2) = \vec{v}_2 . (\vec{v}_3 \wedge \vec{v}_1)$$

2.9 Vector Projection

Let $\vec{v}$ be an arbitrary vector, and Δ be an axis in space defined by its unit vector $\vec{u}$. The orthogonal projection of vector $\vec{v}$ onto axis Δ is the component $\overrightarrow{proj}\,\vec{v}/(\Delta)$ of $\vec{v}$ along this axis.

$$\overrightarrow{proj}\,\vec{v}/(\Delta) = (\vec{v}.\vec{u})\vec{u}$$



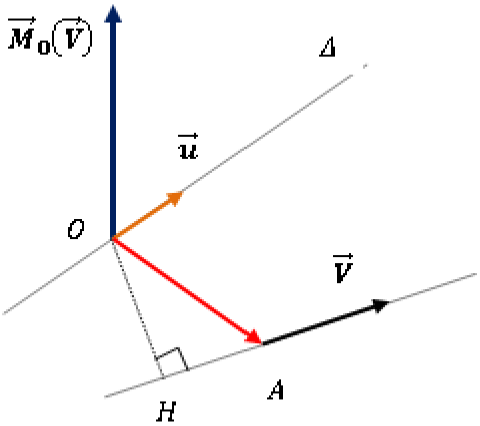
2.10 Vector Moment

Consider a vector $\vec{V}$ linked to the point A: $(A, \vec{v})$.

2.10.1 Vector Moment of $\vec{V}$ with Respect to a Point O

By definition, the vector moment $\vec{M}_0$ with respect to O is given by:

$$\vec{M}_0(\vec{V}) = \overrightarrow{OA} \wedge \vec{V}$$



2.10.2 Moment of $\vec{V}$ with Respect to the Axis Δ Carrying the Unit Vector $\vec{u}$

By definition, the moment of $\vec{V}$ with respect to the axis Δ is given by:

$$\vec{M}_\Delta(\vec{V}) = \vec{u}.\vec{M}_0(\vec{V})$$

This moment is a scalar quantity. It is independent of the choice of O on the axis. In fact, if $O' \in \Delta$,

$$\vec{u}.\vec{M}_0(\vec{V}) = \vec{u}.\vec{M}_{O'}(\vec{V})$$

2.11 Operators and Vectors:

2.11.1 Gradient Operator in an Orthonormal Coordinate System $R(O,\vec{\imath},\vec{\jmath},\vec{k})$

The gradient operator, denoted as $\vec{\nabla}$, is defined as:

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}$$

It represents the derivative in space along the three directions of the unit vectors. The gradient of a scalar quantity U is defined as the vector derivative along the respective directions $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$ with respect to the variables x, y, z.

$$\overrightarrow{grad}\mathrm{U}(x,y,z) = \frac{\partial U}{\partial x}\vec{\imath} + \frac{\partial U}{\partial y}\vec{\jmath} + \frac{\partial U}{\partial z}\vec{k} \quad ou \qquad \overrightarrow{grad}\mathrm{U} = \vec{\nabla}U$$

The gradient of a scalar function provides information about the rate and direction of the most rapid change of the scalar quantity at a given point in space.

2.11.2 Divergence operator in an orthonormal coordinate system $R(O,\vec{\imath},\vec{\jmath},\vec{k})$

The divergence is defined as follows:

Given a vector $\vec{V} = V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{k}$, the divergence of $\vec{V}$, denoted as $div\vec{V}$ or $\vec{\nabla}.\vec{V}$, is calculated as:

$$div\vec{V} = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right).V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{k} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Here, $\frac{\partial V_x}{\partial x}$, $\frac{\partial V_y}{\partial y}$ and $\frac{\partial V_z}{\partial z}$ represent the partial derivatives of the vector components V_x, V_y, and V_z with respect to their corresponding coordinate variables x, y and z.

The divergence of a vector is a scalar quantity. It indicates the rate at which the vector field "diverges" or spreads out from a given point in space. It provides information about the behavior of the vector field in terms of expansion or contraction at different points.

2.11.3 Curl operator in an orthonormal coordinate system $R(O,\vec{\imath},\vec{\jmath},\vec{k})$

In an orthonormal coordinate system $R(O,\vec{\imath},\vec{\jmath},\vec{k})$, the curl (or rotational) operator is defined as the cross product of the operator $\vec{\nabla} = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}$ with the vector $\vec{V}$.

The curl of $\vec{V}$, denoted as curl$\vec{V}$ or $\vec{\nabla} \wedge \vec{V}$, is calculated as:

$$curl\vec{V} = \overrightarrow{rot}\vec{V} = \vec{\nabla} \wedge \vec{V}; \qquad curl\vec{V} = \overrightarrow{rot}\vec{V} = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right) \wedge V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{k}$$

Here, $\frac{\partial V_x}{\partial x}$, $\frac{\partial V_y}{\partial y}$ and $\frac{\partial V_z}{\partial z}$ represent the partial derivatives of the vector components V_x, V_y, and V_z with respect to their corresponding coordinate variables x, y and z.

The curl of a vector is also a vector. It indicates the rotational behavior of the vector field around a given point in space. It provides information about the circulation or rotation of the vector field at different points.

In matrix form, the curl can be expressed as follows:

$$\overrightarrow{rot}\vec{V} = \begin{cases} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{cases} \wedge \begin{cases} V_x \\ V_y \\ V_z \end{cases} = \begin{cases} \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \\ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \\ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{cases}$$

Remark:

If f is a scalar field and $\vec{A}$ and $\vec{B}$ *are two arbitrary vectors, the following relations hold true:*

- $div(f\vec{A}) = f div(\vec{A}) + \vec{A}\overrightarrow{grad}f$
- $\overrightarrow{rot}(\overrightarrow{rot}\vec{A}) = \overrightarrow{grad}(div\vec{A}) - \Delta\vec{A}$, avec $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
- $\overrightarrow{rot}(f\vec{A}) = (\overrightarrow{grad}f \wedge \vec{A}) + f\overrightarrow{rot}(\vec{A})$
- $\overrightarrow{rot}(\overrightarrow{grad}f) = \vec{0}$
- $div(\overrightarrow{rot}\vec{A}) = 0$
- $div(\vec{A} \wedge \vec{B}) = \vec{B}.\overrightarrow{rot}(\vec{A}) - \vec{A}.\overrightarrow{rot}(\vec{B})$

Exercises

Dimensional Analysis and Vector Calculus

1. Dimensional Analysis

Exercise 1.1:

Determine the dimensions of energy, power, potential U, and resistance R.

Exercise 1.2:

Determine the dimension of the capacitance C of a capacitor.

Exercise 1.3:

Verify the dimensional consistency of the relationship $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$, which represents the frequency f of oscillation of a solid-spring system. Here, m is the mass of the solid and k is the stiffness constant. The restoring force $\vec{F}$ is related to the elongation l by the equation: $\vec{F} = -k\overrightarrow{\Delta l}$.

Exercise 1.4:

Using the expression for the amount of work exchanged with a perfect gas $\delta W = -pdV$ and the work done by a force, retrieve the dimension of pressure.

Exercise 1.5:

The vibration frequency of a water droplet depends on several parameters. We assume that surface tension is the predominant factor in the cohesion of the droplet. Therefore, the factors involved in the expression of the vibration frequency f are:

- R, the radius of the droplet
- ρ , the density, to account for inertia
- A, the constant appearing in the expression of the force due to surface tension (the dimension of A is force per unit length)

We can write: $f = k_1 R^a \rho^b A^c$, where k_1 is a dimensionless constant; ; a, b and c are the exponents of R, ρ and A, respectively.

Determine the values of a, b and c.

Exercise 1.6:

In a fluid, a sphere with radius r moving at velocity v is subjected to a frictional force given by $F = -6\pi\eta r v$, where η is the fluid viscosity.

1. What is the dimension of η?
2. When the sphere is released without any initial velocity at time $t = 0$, its velocity for $t > 0$ is given by $v = a\left(1 - exp(-t/b)\right)$, where a and b are two quantities that depend on the characteristics of the fluid. What are the dimensions of a and b?
3. If ρ represents the fluid density, find a simple dimensionless combination $R_e = \rho^\alpha v^\beta r^\gamma \eta^\delta$ (taking $\alpha = 1$ among the various possible choices). This yields the Reynolds number, which characterizes the flow regime of a fluid (laminar or turbulent).

Solution

Exercise 1.1:

- **Energy:**

Starting from the relation $E_c = \frac{1}{2}mv^2$, we first obtain:

$[E_c] = [m][v]^2$

Furthermore, since velocity $v = \frac{dx}{dt}$ is expressed in $m.s^{-1}$, we have $[v] = LT^{-1}$. Therefore, $[E_c] = ML^2T^{-2}$, and the unit is Joule.

There is a multiple way to resolve this issue, for example:

The dimensions of energy can be obtained from its definition as the ability to do work. Work is defined as force multiplied by distance. Therefore, the dimensions of energy are $[E] = [force]$x$[distance]$. Using the fundamental dimensions, we have $[E] = ML^2T^{-2}$.

- **Power (P):**

Power is defined as the rate at which work is done or energy is transferred per unit of time. Mathematically, power (P) is given by $P = E/t$, where E represents energy and t represents time.

Using the dimensions of energy $[E] = ML^2T^{-2}$ derived earlier, and the dimension of time $[T]$, we can determine the dimensions of power as follows:

$$[P] = \frac{[E]}{[t]} = \frac{ML^2T^{-2}}{[t]} = ML^2T^{-3}$$

Therefore, the dimensions of power are $[P] = ML^2T^{-3}$, and the unit of power in the International System of Units (SI) is the Watt (W).

- **Potential (U):**

The potential is related to power (P) and current (I) through the equation $P = U\,I$, where U represents potential (voltage) and I represents current.

By analyzing the dimensions of power $[P] = [U][I]$, $[P] = ML^2T^{-3}$ derived earlier, and the dimensions of current $[I] = [A]$ [I], we can determine the dimensions of potential as follows:

$[P] = [U][I]$, after that $[U] = [P][I]^{-1} = ML^2T^{-3}I^{-1}$

Therefore, the dimensions of potential are $[U] = [P][I]^{-1} = ML^2T^{-3}I^{-1}$. In the International System of Units (SI), the unit of potential is the Volt (V).

- **Resistance (R):**

Resistance is a property of electrical circuits and is measured in ohms (Ω). The dimensions of resistance can be derived from Ohm's law, which states that resistance is equal to voltage divided by current $(R = V/I)$. Therefore, the dimensions of resistance are:

$[R] = [U][I]^{-1} = ML^2T^{-3}I^{-2}$, where [*V*] represents voltage and [*I*] represents current.

Exercise 1.2:

L'énergie emmagasinée par le condensateur est $E_c = \frac{1}{2}\,CU_c^2$.

Avec l'exercice précédent, on sait que $[E_c] = ML^2T^{-2}$ et $[U_c] = ML^2T^{-3}I^{-1}$

On en déduit: $[C] = \frac{[E_c]}{[U_c]^2} = M^{-1}L^{-2}T^4I^2$.

The energy stored in a capacitor is given by the equation $E_c = \frac{1}{2}\,CU_c^2$ represents energy, C represents capacitance, and U_c represents voltage.

From the previous exercise, we know that $[E_c] = ML^2T^{-2}$ and $[U_c] = ML^2T^{-3}I^{-1}$.

We can determine the dimensions of capacitance by dividing the dimensions of energy by the square of the dimensions of voltage:

$$[C] = \frac{[E_c]}{[U_c]^2} = M^{-1}L^{-2}T^4I^2$$

Therefore, the dimensions of capacitance are $[C] = M^{-1}L^{-2}T^4I^2$. In the International System of Units (*SI*), the unit of capacitance is the Farad (*F*).

Exercise 1.3:

The goal is to verify that both sides of the equation have the same dimensions.

Since f represents a frequency, its dimension is $[f] = T^{-1}$.

Let's determine the dimension of the right-hand side. We have $\left[\frac{1}{2\pi}\right] = 1$ and $[m] = M$. From the equation $\vec{F} = -k\overrightarrow{\Delta l}$, we deduce that $[F] = [k]L$. Since $[F] = MLT^{-2}$, we can conclude that $[k] = MT^{-2}$. Therefore,

$$\left[\frac{1}{2\pi}\sqrt{\frac{k}{m}}\right] = \sqrt{\frac{[k]}{[m]}} = \sqrt{\frac{MT^{-2}}{M}} = T^{-1}.$$

Both sides have the same dimension: the equation is homogeneous.

Exercise 1.4:

Using the given formula, we have$[\delta W] = [p] \times [dV]$.

Sine: $[\delta W] = [F]L = MLT^{-2}L = ML^2T^{-2}$ et $[dV] = L^3$, we can deduce:

$$[p] = ML^{-1}T^{-2}$$

The unit is Pascal.

Exercise 1.5:

To determine the dimensions of the variables in the equation $f = k_1 R^a \rho^b A^c$, we can equate the dimensions on both sides of the equation.

On the left-hand side, $[f] = T^{-1}$ (dimension of time).

On the right-hand side, we have the following dimensions:

$$\left.\begin{array}{l} [R] = [dimension\ of\ length] = L \\ [\rho] = [dimension\ of\ mass/V] = ML^{-3} \\ [A] = \left[dimension\ of \frac{force}{L}\right] = MLT^{-2}/L \\ \quad = MT^{-2} \\ [f] = [frequence] = T^{-1} = [fk_1R^a\rho^bA^c] \end{array}\right\} \Longrightarrow \quad T^{-1} = L^a(ML^{-3})^b(MT^{-2})^c$$

$$\left\{\begin{array}{l} T^{-1} = T^{-2c} \\ 1 = L^aL^{-3b} \\ 1 = M^bM^c \end{array}\right.$$

We can deduct from the previous equality :

$$\begin{cases} 2c & = 1 \\ a - 3b & = 0 \\ b + c & = 0 \end{cases} \Longrightarrow \begin{cases} a = -3/2 \\ b = -1/2 \\ c = 1/2 \end{cases} \Longrightarrow \boxed{f = k_1 \frac{1}{R}\sqrt{\frac{A}{R\,\rho}}}$$

Please note that k_1 is a dimensionless constant and does not affect the dimensions in this case.

Exercice 1.6:

To make the expression dimensionless, we need to choose the values of $a, \beta,$ and δ such that all the dimensions cancel out. Let's determine the values of these exponents:

1)

$$F = -6\pi\eta r v \Longrightarrow \eta = \frac{F}{6\pi r v} = \frac{[masse].[accélération]}{[r].[v]} = \frac{M.(L.T^{-2})}{L.(L.T^{-1})}$$

$$\Longrightarrow [\eta] = \boxed{M.L^{-1}}$$

2) The argument of the exponential function is dimensionless, so $[b] = T$. The right-hand side of the equation represents a velocity, so $[a] = L.T^{-1}$.

3)

$$R_e = \rho^\alpha v^\beta r^\gamma \eta^\delta \Longrightarrow R_e = [\rho]^\alpha [v]^\beta [r]^\gamma [\eta]^\delta = (M.L^{-3})^\alpha (L.T^{-1})^\beta L^\gamma (M.L.T)^\delta$$

$$\Longrightarrow R_e = M^{\alpha+\delta}.L^{-3\alpha+\beta+\gamma-\delta}.T^{-\beta-\delta}$$

Since R_e is dimensionless according to the statement, we can set $\alpha = 1$, we conclude:

$$\begin{cases} 1 + \delta = 0 \\ -3\alpha + \beta + \gamma - \delta = 0 \\ -\beta - \delta = 0 \end{cases} \Longrightarrow \begin{cases} \delta = -1 \\ \beta = 1 \\ \gamma = 1 \end{cases}$$

In conclusion, the values of the exponents for the variables in the expression are:

$$R_e = \frac{\rho v r}{\eta} \frac{\rho^\alpha v \quad r}{\eta}$$

2. Vector Calculus

Exercise 2.1:

Two points A and B have Cartesian coordinates $R\,(OXYZ)$ in space: $A(2,3,-3)$, $B(5,7,2)$

1. Determine the components of vector $\overrightarrow{AB}$, as well as its magnitude, direction, and sense.

Exercise 2.2:

In an orthonormal coordinate system $R\ (OXYZ)$, we consider the three vectors:

$\overrightarrow{V_1} = 3\vec{\imath} - 4\vec{\jmath} + 4\vec{k}$, $\overrightarrow{V_2} = 2\vec{\imath} + 3\vec{\jmath} - 4\vec{k}$ et $\overrightarrow{V_3} = 5\vec{\imath} - \vec{\jmath} + 3\vec{k}$.

a. Calculate the magnitudes of $\overrightarrow{V_1}$, $\overrightarrow{V_2}$ et $\overrightarrow{V_3}$.

b. Calculate the components and magnitudes of the vectors:

$\vec{A} = \overrightarrow{V_1} + \overrightarrow{V_2} + \overrightarrow{V_3}$ et $\vec{B} = 2\overrightarrow{V_1} - \overrightarrow{V_2} + \overrightarrow{V_3}$.

c. Determine the unit vector carried by $\vec{C} = \overrightarrow{V_1} + \overrightarrow{V_3}$.

d. Calculate the dot product $\overrightarrow{V_1}.\overrightarrow{V_3}$ d determine the angle formed by the two vectors.

e. Calculate the cross product $\overrightarrow{V_1} \wedge \overrightarrow{V_3}$.

Exercise 2.3:

1. Show that the area of a parallelogram is given by $|\vec{A} \wedge \vec{B}|$, where $|\vec{A}|$ and $|\vec{B}|$ are the lengths of the sides of the parallelogram formed by the two vectors.
2. Prove that vectors A⃗ and B⃗ are perpendicular if $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$.

Exercise 2.4:

Given the two vectors $\vec{A} = \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}$, $\vec{B} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$

Find the values of α and β such that $\vec{B}$ is parallel to $\vec{A}$. Then determine the unit vector for each of the two vectors.

Exercise 2.5:

Find the sum of the following three vectors:

$\overrightarrow{V_1} = 5\vec{\imath} - 2\vec{\jmath} + 2\vec{k}$, $\overrightarrow{V_2} = -3\vec{\imath} + \vec{\jmath} - 7\vec{k}$ et $\overrightarrow{V_3} = 4\vec{\imath} + 7\vec{\jmath} + 6\vec{k}$.

Calculate the magnitude of the resultant vector and the angles it forms with *OY*, *OX*, and *OZ.*

Exercice 2.6:

Given the following vectors :

$\overrightarrow{U_1} = A_1\vec{\imath} + A_2\vec{\jmath} + A_3\vec{k}$ and $\overrightarrow{U_2} = B_1\vec{\imath} + B_2\vec{\jmath} + B_3\vec{k}$

a. Calculate the dot products: $\overrightarrow{U_1}.\overrightarrow{U_2}$, $\overrightarrow{U_1} \cdot \overrightarrow{U_1}$, $\overrightarrow{U_2}.\overrightarrow{U_2}$,

Given: $\overrightarrow{V_1} = 2\vec{\imath} - \vec{\jmath} + 5\vec{k}$, $\overrightarrow{V_2} = -3\vec{\imath} + 1.5\vec{\jmath} - 7.5\vec{k}$ et $\overrightarrow{V_3} = -5\vec{\imath} + 4\vec{\jmath} + \vec{k}$

b. Calculate : $\overrightarrow{V_1}.\overrightarrow{V_2}$ et $\overrightarrow{V_1} \wedge \overrightarrow{V_2}$;

c. Without making a graphical representation, what can we say about the direction and sense of vector $\overrightarrow{V_2}$ with respect to $\overrightarrow{V_1}$;

d. Calculate the following products : $\overrightarrow{V_1}.(\overrightarrow{V_2} \wedge \overrightarrow{V_3})$ and $\overrightarrow{V_1} \wedge (\overrightarrow{V_2} \wedge \overrightarrow{V_3})$;

e. Determine the area of the triangle formed by vectors $\overrightarrow{V_2}$ et $\overrightarrow{V_3}$

Solution

Exercise 2.1:

The vector $\overrightarrow{AB}$ is given by: $\overrightarrow{AB} = \overrightarrow{OB} + \overrightarrow{OA} = 3\vec{\imath} + 4\vec{\jmath} + 5\vec{k}$

Its magnitude: $|\overrightarrow{AB}| = \|\overrightarrow{AB}\| = AB = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50}$

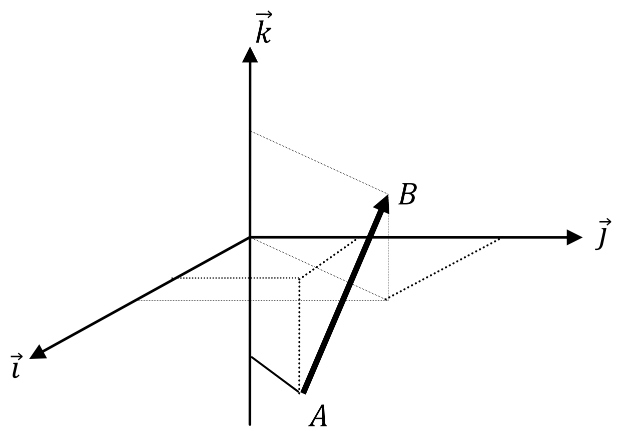
Its direction is determined by the angles (α, β, θ) it makes with each of the coordinate axes. These angles can be found using the dot product of vector $\overrightarrow{AB}$ with the unit vectors of the orthonormal coordinate system:

$$\alpha = (\overrightarrow{AB}, \vec{\imath}): \quad \overrightarrow{AB} \cdot \vec{\imath} = AB.1.\cos\alpha \quad \Leftrightarrow \quad \cos\alpha = \frac{\overrightarrow{AB} \cdot \vec{\imath}}{AB} = \frac{3}{\sqrt{50}} = 0.424 \quad \Longrightarrow \quad \alpha = 64.89°$$

$$\beta = (\overrightarrow{AB}, \vec{\jmath}): \quad \overrightarrow{AB} \cdot \vec{\jmath} = AB.1.\cos\beta \quad \Leftrightarrow \quad \cos\beta = \frac{\overrightarrow{AB} \cdot \vec{\jmath}}{AB} = \frac{4}{\sqrt{50}} = 0.565 \quad \Longrightarrow \quad \alpha = 55.54°$$

$$\theta = (\overrightarrow{AB}, \vec{k}): \quad \overrightarrow{AB} \cdot \vec{k} = AB.1.cos\,\theta \quad \Leftrightarrow \quad cos\,\theta = \frac{\overrightarrow{AB} \cdot \vec{k}}{AB} = \frac{5}{\sqrt{50}} = 0.707 \quad \Longrightarrow \quad \alpha = 44.99°$$

Its sense: Since the dot product of the vector $\overrightarrow{AB}$ with all three unit vectors is positive, it has a positive sense along all three axes of the coordinate system.



Exercise 2.2:

a. Calculating the magnitudes of $\overrightarrow{V_1}$, $\overrightarrow{V_2}$ and $\overrightarrow{V_3}$:

$$\|\overrightarrow{V_1}\| = \sqrt{(3)^2 + (-4)^2 + (4)^2} = \sqrt{9+16+16} \Rightarrow \boxed{\|\overrightarrow{V_1}\| = \sqrt{41} = 6.40}$$

$$\|\overrightarrow{V_2}\| = \sqrt{(2)^2 + (3)^2 + (-4)^2} = \sqrt{4+9+16} \Rightarrow \boxed{\|\overrightarrow{V_2}\| = \sqrt{29} = 5.38}$$

$$\|\overrightarrow{V_3}\| = \sqrt{(5)^2 + (-1)^2 + (3)^2} = \sqrt{25+1+9} \Rightarrow \boxed{\|\overrightarrow{V_3}\| = \sqrt{35} = 5.91}$$

b. Components and magnitudes of $\vec{A}$ and $\vec{B}$:

$$\vec{A} = \overrightarrow{V_1} + \overrightarrow{V_2} + \overrightarrow{V_3} = \begin{cases} 3 \ +2 \ + \ 5 \\ -4+3+(-1) \\ 4+(-4)+3 \end{cases} \Rightarrow \boxed{\vec{A} = 10\vec{\imath} - 2\vec{\jmath} + 3\vec{k}}$$

$$\|\vec{A}\| = \sqrt{(10)^2 + (-2)^2 + (3)^2} = \sqrt{100+4+9} \Rightarrow \boxed{\|\vec{A}\| = \sqrt{113} = 10.63}$$

$$\vec{B} = 2\overrightarrow{V_1} - \overrightarrow{V_2} + \overrightarrow{V_3} = \begin{cases} 2\times 3 \ -2 \ + \ 5 \\ 2\times(-4)-3+(-1) \\ 2\times 4-(-4)+3 \end{cases} \Rightarrow \boxed{\vec{B} = 9\vec{\imath} - 12\vec{\jmath} + 15\vec{k}}$$

$$\|\vec{B}\| = \sqrt{(9)^2 + (-12)^2 + (15)^2} = \sqrt{81+144+225} \Rightarrow \boxed{\|\vec{B}\| = \sqrt{450} = 21.21}$$

c. $\vec{u}_C$?

$$\vec{C} = \overrightarrow{V_1} + \overrightarrow{V_3} = \begin{cases} 3 \ + \ 5 \\ -4+(-1) \\ 4+3 \end{cases} \Rightarrow \vec{C} = 8\vec{\imath} - 5\vec{\jmath} + 7\vec{k}$$

Given that $\vec{C}$ is written as : $\vec{C} = \|\vec{C}\|\vec{u}_C$, First, let's calculate the magnitude of $\vec{C}$.

$$\|\vec{C}\| = \sqrt{(8)^2 + (-5)^2 + (7)^2} = \sqrt{64+25+49} \Rightarrow \|\vec{C}\| = \sqrt{138} = 11.74$$

$$\boxed{\vec{u}_C = \frac{\vec{C}}{\|\vec{C}\|} = \frac{8}{\sqrt{138}}\vec{\imath} - \frac{5}{\sqrt{138}}\vec{\jmath} + \frac{7}{\sqrt{138}}\vec{k}}$$

d. $\overrightarrow{V_1}.\overrightarrow{V_3}$?

$$\overrightarrow{V_1}.\overrightarrow{V_3} = \begin{pmatrix} 3 \\ -4 \\ 4 \end{pmatrix}.\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} = 5 \times 3 + (-4) \times (-1) + 4 \times 3$$

$$\Longrightarrow \quad \boxed{\overrightarrow{V_1}.\overrightarrow{V_3} = 31}$$

Now, let's use the cosine formula for the dot product:

$$\overrightarrow{V_1}.\overrightarrow{V_3} = \|\overrightarrow{V_1}\|\|\overrightarrow{V_3}\| \cos(\overrightarrow{V_1}, \overrightarrow{V_3}) \quad \Longrightarrow$$

$$\cos(\overrightarrow{V_1}, \overrightarrow{V_3}) = \frac{\overrightarrow{V_1}.\overrightarrow{V_3}}{\|\overrightarrow{V_1}\|\|\overrightarrow{V_3}\|}$$

$$\cos(\overrightarrow{V_1}, \overrightarrow{V_3}) = \frac{31}{\sqrt{41}\sqrt{35}}$$

$$\cos(\overrightarrow{V_1}, \overrightarrow{V_3}) = 0.81, \quad \Longrightarrow \quad \boxed{(\overrightarrow{V_1}, \overrightarrow{V_3}) = 35.07°}$$

a. $\overrightarrow{V_1} \wedge \overrightarrow{V_3}$?

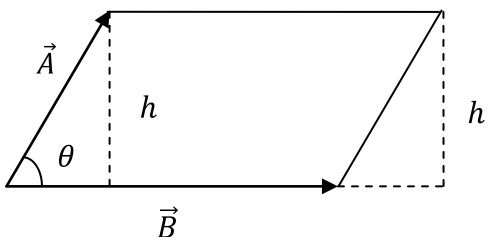
We can use the analytical expression for the cross product:

$$\overrightarrow{V_1} \wedge \overrightarrow{V_3} = \begin{cases} 3 \\ -4 \\ 4 \end{cases} \wedge \begin{cases} 5 \\ -1 \\ 3 \end{cases} = \begin{cases} (-4) \times 3 - (-1) \times 4 \\ 5 \times 4 - 3 \times 3 \\ 3 \times (-1) - 5 \times (-4) \end{cases}$$

$$\Longrightarrow \boxed{\overrightarrow{V_1} \wedge \overrightarrow{V_3} = -8\vec{\imath} + 11\vec{\jmath} + 17\vec{k}}$$

Exercice 2.3:

a. The surface area of a parallelogram is given by: $S = h.|\vec{B}|$



According to the figure, we can write:

$$h = |\vec{A}| \sin\theta$$

By substituting the value of h into the first equation, the surface area becomes:

$$S = |\vec{A}|\,|\vec{B}| \sin\theta$$

This is nothing but the analytical expression of the cross product:

$$S = |\vec{A} \wedge \vec{B}| = |\vec{A}|\,|\vec{B}| \sin\theta$$

Similarly, we can show that the surface area of a triangle with sides :$|\vec{A}|$ and $|\vec{B}|$ is equal to:

$$S_0 = \frac{1}{2}|\vec{A} \wedge \vec{B}| = \frac{1}{2}|\vec{A}|\,|\vec{B}|\sin\theta$$

Consider the two vectors:

$$\vec{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \quad ; \quad \vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}$$

$$|\vec{A} + \vec{B}| = \left[(A_x + B_x)^2 + \left(A_y + B_y\right)^2 + (A_z + B_z)^2\right]^{1/2}$$

$$|\vec{A} - \vec{B}| = \left[(A_x - B_x)^2 + \left(A_y - B_y\right)^2 + (A_z - B_z)^2\right]^{1/2}$$

By equating these two expressions and expanding, we arrive at the result:

$$|\vec{A} + \vec{B}| = A_x B_x + A_y B_y + A_z B_z = 0$$

which is nothing but the dot product $(\vec{A}.\vec{B}) = 0$, indicating that $\vec{A}$ is orthogonal (perpendicular) to $\vec{B}$:

$$(\vec{A}.\vec{B}) = 0 \quad \Leftrightarrow \quad \vec{A} \perp \vec{B}.$$

Exercise 2.4:

For the two vectors $\vec{A}$ and $\vec{B}$ to be parallel, it is necessary that $\vec{B} = \lambda\vec{A}$, where λ is a constant.

Starting from this, we can write:

$$\frac{\vec{B}}{\lambda} = \vec{A} \quad \Rightarrow \quad \frac{\vec{B}}{\lambda} = \begin{pmatrix} 2/\lambda \\ -3/\lambda \\ 4/\lambda \end{pmatrix} = \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}$$

We deduce the value and subsequently the values of α and β:

$$2/\lambda = 1 \Rightarrow \boxed{\lambda = 2}$$

$$-3/\lambda = \alpha \Rightarrow \boxed{\alpha = -1.5}$$

$$4/\lambda = \beta \Rightarrow \boxed{\beta = 2}$$

$$\Rightarrow \vec{B} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \; ; \; \vec{A} = \begin{pmatrix} 1 \\ -1.5 \\ 2 \end{pmatrix}$$

We verify the two results by calculating $\vec{A} \wedge \vec{B} = \vec{0}$.

The unit vectors corresponding to each of the two vectors $\vec{A}$ and $\vec{B}$ are:

$$\vec{A} = \vec{\imath} - 1.5\vec{\jmath} + 2\vec{k}; \quad \frac{\vec{A}}{A} = \vec{u}_A \quad \Longrightarrow \quad \vec{u}_A = \frac{1}{\sqrt{7.25}}\vec{\imath} - \frac{1.5}{\sqrt{7.25}}\vec{\jmath} + \frac{2}{\sqrt{7.25}}\vec{k}$$

$$\vec{B} = 2\vec{\imath} - 3\vec{\jmath} + 4\vec{k}; \quad \frac{\vec{B}}{B} = \vec{u}_B \quad \Longrightarrow \quad \vec{u}_B = \frac{2}{\sqrt{29}}\vec{\imath} - \frac{3}{\sqrt{29}}\vec{\jmath} + \frac{4}{\sqrt{29}}\vec{k}$$

Exercise 2.5:

The resultant vector $\vec{R}$ of the three vectors can be written as follows:

$$\vec{R} = \vec{V_1} + \vec{V_2} + \vec{V_3}$$

$$\vec{R} = \vec{V_1} + \vec{V_2} + \vec{V_3} = \begin{cases} 5 \ + (-3) \ + \ 4 \\ -2 + 1 + 7 \\ 2 + (-7) + 6 \end{cases} \qquad \Longrightarrow \qquad \boxed{\vec{A} = 6\vec{\imath} + 6\vec{\jmath} + \vec{k}}$$

$$\|\vec{R}\| = \sqrt{(6)^2 + (6)^2 + (1)^2} = \sqrt{73} \Longrightarrow \boxed{\|\vec{R}\| = 8.54}$$

$R_x = R\cos(\vec{R}, OX)$, $cos(\vec{R}, OX) = R_x/R = 6/8.54$, $cos(\vec{R}, OX) = 0.7 \Rightarrow (\vec{R}, OX) \simeq 45.6°$

$R_y = R\cos(\vec{R}, OY)$, $cos(\vec{R}, OY) = R_y/R = 6/8.54$, $cos(\vec{R}, OY) = 0.7 \Rightarrow (\vec{R}, OY) \simeq 45.6°$

$R_z = R\cos(\vec{R}, OZ)$, $cos(\vec{R}, OZ) = R_z/R = 1/8.54$, $cos(\vec{R}, OZ) = 0.12 \Rightarrow (\vec{R}, OZ) \simeq$ 83.1°

Exercice 2.6:

a. $\vec{U_1}.\vec{U_2} = A_1\,B_1 + A_2B_2 + A_3B_3$

$$\vec{U_1} \cdot \vec{U_1} = {A_1}^2 + {A_2}^2 + {A_3}^2$$

$$\vec{U_2}.\vec{U_2} = {B_1}^2 + {B_2}^2 + {B_3}^2$$

b. $\vec{V_1}.\vec{V_2} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}\begin{pmatrix} -3 \\ 1.5 \\ -7.5 \end{pmatrix} = -6 - 1.5 - 37.5 = \boxed{-45}$

$$\overrightarrow{V_1} \wedge \overrightarrow{V_2} = \begin{cases} 2 \\ -1.5 \\ 5 \end{cases} \wedge \begin{cases} -3 \\ 1.5 \\ -7.5 \end{cases} = \begin{cases} 7.5 - 7.5 \\ -1.5 - 1.5 \\ 3 - 3 \end{cases} = \begin{cases} 0 \\ 0 \\ 0 \end{cases}$$

c. Since the cross product of the two vectors is zero, they are parallel.

$$\overrightarrow{V_1} \wedge \overrightarrow{V_2} = \vec{0} \quad \Rightarrow \quad \overrightarrow{V_1} \parallel \overrightarrow{V_2}$$

Furthermore, the dot product is negative, $\overrightarrow{V_1}.\overrightarrow{V_2} = -45$, so the vectors $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$ are parallel and have opposite directions.

c. $$\overrightarrow{V_1}.\left(\overrightarrow{V_2} \wedge \overrightarrow{V_3}\right) = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}.\left(\begin{cases} -3 \\ 1.5 \\ -7.5 \end{cases} \wedge \begin{cases} -5 \\ 4 \\ 1 \end{cases}\right) = \begin{cases} 2 \\ -1 \\ 5 \end{cases}.\begin{cases} 31.5 \\ 4 \\ 1 \end{cases} = 63 - 40.5 - 22.5 = \boxed{0}$$

d. $$\overrightarrow{V_1} \wedge \left(\overrightarrow{V_2} \wedge \overrightarrow{V_3}\right) = \begin{cases} 2 \\ -1 \\ 5 \end{cases} \wedge \left(\begin{cases} -3 \\ 1.5 \\ -7.5 \end{cases} \wedge \begin{cases} -5 \\ 4 \\ 1 \end{cases}\right) = \begin{cases} 2 \\ -1 \\ 5 \end{cases} \wedge \begin{cases} 31.5 \\ 40.5 \\ -4.5 \end{cases} = \begin{cases} -198 \\ 166.5 \\ 112.5 \end{cases}$$

$$\overrightarrow{V_1} \wedge \left(\overrightarrow{V_2} \wedge \overrightarrow{V_3}\right) = \boxed{-198\vec{\imath} + 166.5\vec{\jmath} + 112.5\vec{k}}$$

f. La surface du triangle formé par les vecteurs $\overrightarrow{V_2}$ et $\overrightarrow{V_3}$ est donnée par la moitié du module du produit vectoriel des deux vecteurs:

Nous avons:

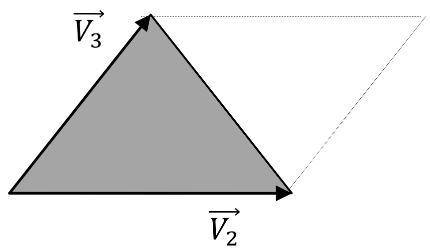
The surface area of the triangle formed by the vectors $\overrightarrow{V_2}$ and $\overrightarrow{V_3}$ is half the magnitude of the cross product of the two vectors.

We have:

$\overrightarrow{V_2} \wedge \overrightarrow{V_3} = 31.5\vec{\imath} + 40.5\vec{\jmath} - 4.5\vec{k}$ alors:

$$\left|\overrightarrow{V_2} \wedge \overrightarrow{V_3}\right| = \sqrt{(31.5)^2 + (40.5)^2 + (-4.5)^2} = 51.50$$

$$S = \frac{\left|\overrightarrow{V_2} \wedge \overrightarrow{V_3}\right|}{2} = \frac{51.50}{2} = \boxed{25.75}$$



It is half the surface area of the parallelogram.

Chapitre II

Kinematics

II.1 *Kinematics*

2.1 Introduction

Kinematics: It is the study of motion independent of the causes that produce it. *Particle:* A mathematical model that associates a geometric point with the total mass of an object.

2.2 Locating a Particle in Space and Time

a. Locating in Space

To locate a point in space, we need a known spatial reference frame characterized by an origin and a right-handed orthonormal basis (coordinate system).

b. Locating in Time

To locate a moving particle, in addition to the spatial reference frame, we need a time reference frame characterized by a time origin and a clock to measure the passage of time.

c. Reference Frame

The combination of a spatial reference frame and a time reference frame.

d. Trajectory

We describe the motion of a particle by knowing its position in space at each instant. The collection of these positions constitutes the trajectory.

Thus, the trajectory is the path described by the particle in the reference frame of study.

Notation: The reference frame, with the spatial reference frame having the origin O and the basis vectors

$$R = (O, \overrightarrow{e_x}, \overrightarrow{e_y}, \overrightarrow{e_z})$$

2.3 Coordinate System

2.3.1 Cartesian Coordinates

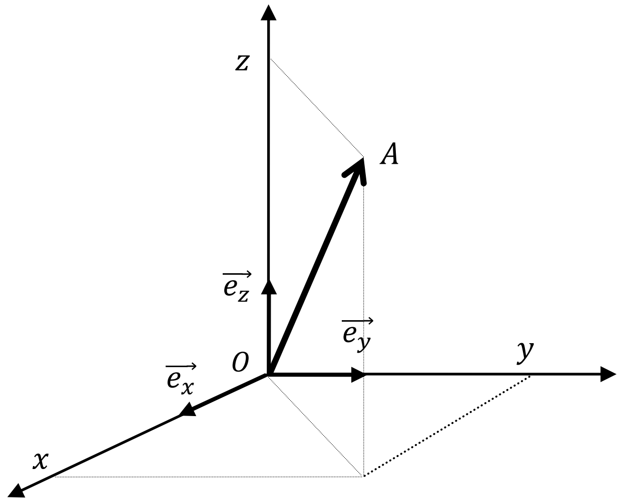
- By definition, the Cartesian coordinates of a point A are (x, y, z).

- The Cartesian basis is the orthonormal direct basis often denoted by $(\vec{e_x}, \vec{e_y}, \vec{e_z})$ or $(\vec{i}, \vec{j}, \vec{k})$
- The position vector:

$$\overrightarrow{OA} = x\vec{e_x} + y\vec{e_y} + z\vec{e_z}$$

- The infinitesimal displacement:

$$d\overrightarrow{OA} = dx\vec{e_x} + dy\vec{e_y} + dz\vec{e_z}$$



2.3.2 Cylindrical coordinates

By definition, the cylindrical coordinates of a point A are:

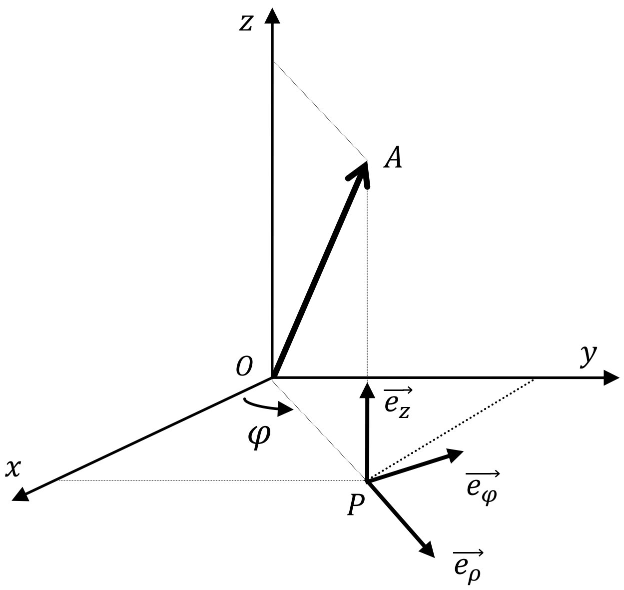
(ρ, φ, z) where $\rho \geq 0$ and $0 \leq \varphi \leq 2\pi$

- The local cylindrical base is the orthonormal direct base: $(\vec{e_\rho}, \vec{e_\varphi}, \vec{e_z})$ such that :

  $\vec{e_\rho} = \frac{\overrightarrow{OP}}{\rho}$ $\left(\rho = \left|\overrightarrow{OP}\right|\right)$; $\vec{e_\varphi}$ ⊥ perpendicular to the plane defined by φ and is defined as:

  $\vec{e_\varphi} = \vec{e_z} \wedge \vec{e_\rho}$

  Relations between Cartesian coordinates and cylindrical coordinates:



- *Radial coordinate* (ρ)*:* ρ is the distance from the point to the origin in the $x\,y$ plane and can be calculated from the Cartesian coordinates as follows: $\rho = \sqrt{x^2 + y^2}$
- *Azimuthal coordinate* (φ)*:* φ is the angle formed between the positive x-axis and the projection of the position vector onto the $x\,y$ plane. It can be calculated from the Cartesian coordinates as follows: $\varphi = arctang(y/x)$.

- *Vertical coordinate* (z)*:* z remains unchanged and corresponds to the Cartesian coordinate z.

Conversely, Cartesian coordinates can be expressed in terms of cylindrical coordinates as follows:

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases}$$

Coordinate z: z remains unchanged.

- The position vector:

$$\overrightarrow{OA} = \rho\overrightarrow{e_\rho} + z\overrightarrow{e_z}$$

- The infinitesimal displacement:

$$d\overrightarrow{OA} = d\rho\overrightarrow{e_\rho} + \rho d\varphi\overrightarrow{e_\varphi} + dz\overrightarrow{e_z}$$

2.3.3 2.3.3 Polar Coordinates

The polar coordinates of a point A are (ρ,φ) where $\rho \geq 0$ et $0 \leq \varphi \leq 2\pi$

The local polar basis is defined as:

$$\overrightarrow{e_\rho} = \frac{\overrightarrow{OA}}{\rho} \quad (\rho = |\overrightarrow{OA}|)\ ;\ \overrightarrow{e_\varphi} \perp \overrightarrow{e_\rho}$$

2.3.4 Spherical Coordinates

- The spherical coordinates of point A are: (r,θ,φ), where $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$

  The spherical basis is: $(\overrightarrow{e_r}, \overrightarrow{e_\theta}, \overrightarrow{e_\varphi})$ where:

$$\overrightarrow{e_r} = \frac{\overrightarrow{OA}}{r}\ ;\ \overrightarrow{e_\theta} \perp \overrightarrow{e_r} \text{ et } \overrightarrow{e_\varphi} = \overrightarrow{e_r} \wedge \overrightarrow{e_\theta}$$

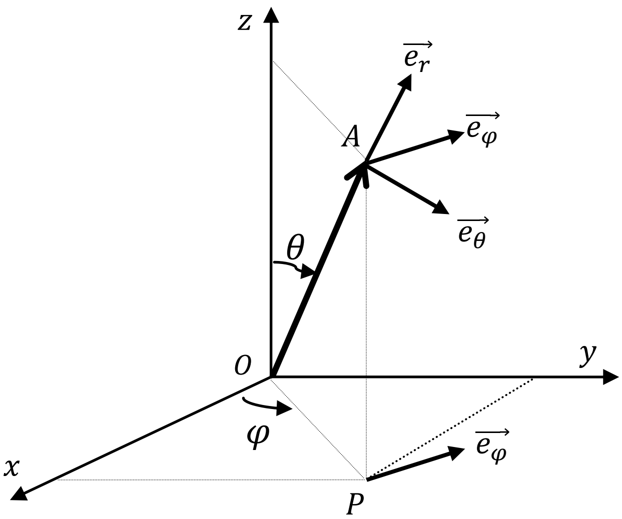
- The position vector is : $\overrightarrow{OA} = r\overrightarrow{e_r}$

Relations between spherical coordinates and Cartesian coordinates:

$$\begin{cases} x = r\sin\theta\sin\varphi \\ y = r\sin\theta\cos\varphi \\ z = r\cos\theta \end{cases}$$

- The infinitesimal displacement:

$$d\overrightarrow{OA} = dr\overrightarrow{e_r} + rd\theta\overrightarrow{e_\theta} + r\sin\theta\, d\varphi\overrightarrow{e_\varphi}$$



2.4 2.4 Kinematics of a Particle

2.4.1 Velocity Vector

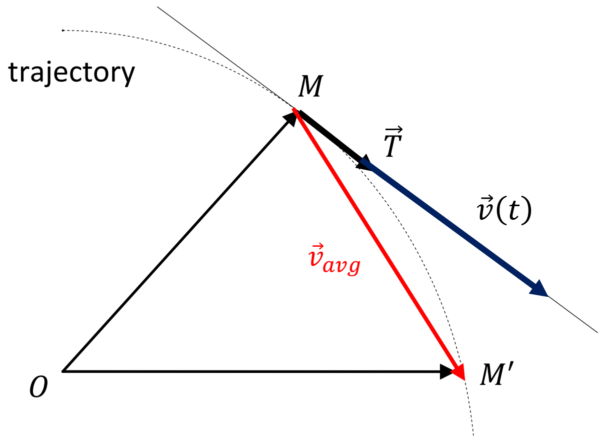
Velocity is defined as the distance traveled per unit of time.

2.4.1.1 Average Velocity Vector

Referring to the Figure below: between the instant t when the particle occupies position M, and the instant t' when the particle occupies position M', the average velocity vector is defined as the expression:

$$\vec{v}_{avg} = \frac{\overrightarrow{MM'}}{t'-t} \quad ; \quad v_{avg} = \frac{\left|\overrightarrow{MM'}\right|}{\Delta t}$$

$\overrightarrow{MM'}$ called the displacement vector.

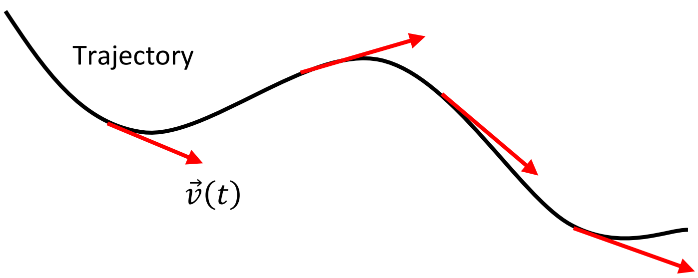


2.4.1.2 Instantaneous velocity vector

The instantaneous velocity vector, which is closely related to the time *t*, is the derivative of the position vector with respect to time.

$$\vec{v}(t) = \lim_{t \to t'} \frac{\overrightarrow{OM'} - \overrightarrow{OM}}{t - t'} = \lim_{t \to t'} \frac{\Delta\overrightarrow{OM}}{\Delta t} = \frac{d\overrightarrow{OM}}{dt}$$

$$\vec{v}(t) = \frac{d\overrightarrow{OM}}{dt}$$



NB: The instantaneous velocity vector $\vec{v}(t)$ is tangent to the trajectory at point M; it is always oriented in the direction of motion.

Convention:

Newton's Notation: The derivative with respect to time is denoted by placing a dot above the symbol of the variable. If the derivative is with respect to a variable other than time, the notation is to put an apostrophe (') after the symbol of the variable to be differentiated.

Leibnitz's Notation: The derivative of y, for example, with respect to time, is denoted as dy/dt. Thus, we can write:

$$\dot{x} = \frac{dx}{dt} \ ; \quad \dot{y} = \frac{dy}{dt} \ ; \quad \dot{z} = \frac{dz}{dt}$$

2.4.1.3 Magnitude of Instantaneous Velocity Vector

The magnitude of the velocity vector is written as follows:

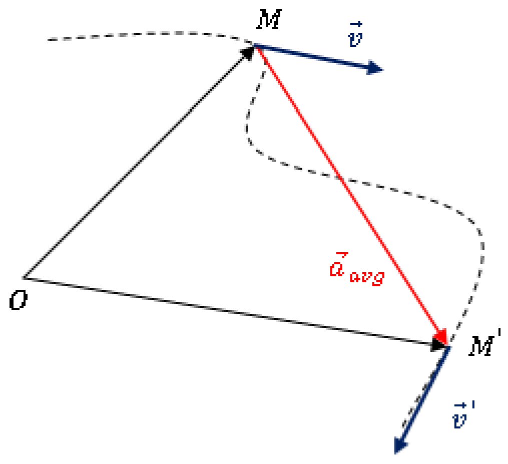
$$v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

The unit of velocity in the International System of Units (SI) is m/s or $m.s^{-1}$. The components of the vector $\overrightarrow{OM}$ and v in Cartesian coordinates are therefore:

$$\overrightarrow{OM} = \ _R\begin{cases} x \\ y \\ z \end{cases} \rightarrow \quad \vec{v} = \ _R\begin{cases} \dot{x} = v_x \\ \dot{y} = v_y \\ \dot{z} = v_z \end{cases}$$

2.4.2 Acceleration Vector

Acceleration is defined as the change in velocity per unit of time.



2.4.2.1 Average Acceleration Vector

Considering two different instants t and t' corresponding to the position vectors $\overrightarrow{OM}$ and $\overrightarrow{OM'}$ and the instantaneous velocity vectors $\vec{v}$ et $\vec{v'}$, the average acceleration vector is defined by the expression::

$$\vec{a}_{avg} = \frac{\vec{v'} - \vec{v}}{t' - t} = \frac{\Delta\vec{v}}{\Delta t} \quad ; \quad a_{avg} = \frac{|\Delta\vec{v}|}{\Delta t}$$

2.4.2.2 Instantaneous Acceleration Vector

The instantaneous acceleration vector of a motion is defined as the derivative of the instantaneous velocity vector with respect to time.

$$\vec{a}(t) = \lim_{t' \to t} \frac{\vec{v}' - \vec{v}}{t' - t} = \lim_{t' \to t} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2}$$

Here are the expressions of position, velocity, and acceleration vectors in Cartesian coordinates, using the conventions of Newton and Leibnitz:

Position vector:

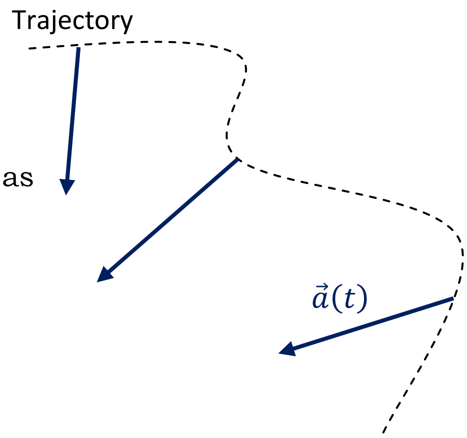
$$\boxed{\overrightarrow{OM} = \vec{r} = x.\vec{\imath} + y.\vec{\jmath} + z.\vec{k} \implies \vec{v} = \dot{x}.\vec{\imath} + \dot{y}.\vec{\jmath} + \dot{z}.\vec{k} \implies \vec{a} = \ddot{x}.\vec{\imath} + \ddot{y}.\vec{\jmath} + \ddot{z}.\vec{k}}$$

Velocity vector:

$$\boxed{\vec{v}(t) = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k} \implies \vec{v}(t) = \frac{d^2x}{dt^2}\vec{\imath} + \frac{d^2y}{dt^2}\vec{\jmath} + \frac{d^2z}{dt^2}\vec{k}}$$

NB: The acceleration vector is always directed towards the concave part of the trajectory.

2.4.2.3 2.4.2.3 Magnitude of the instantaneous acceleration vector

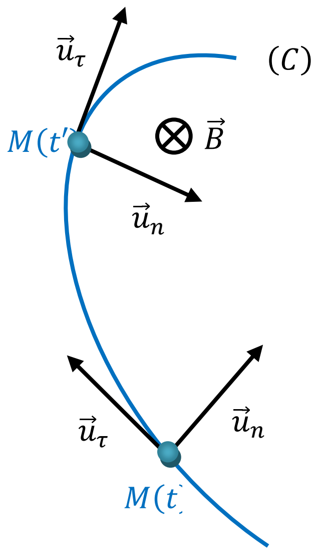


The magnitude of the acceleration vector is written as follows:

$$a = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}$$

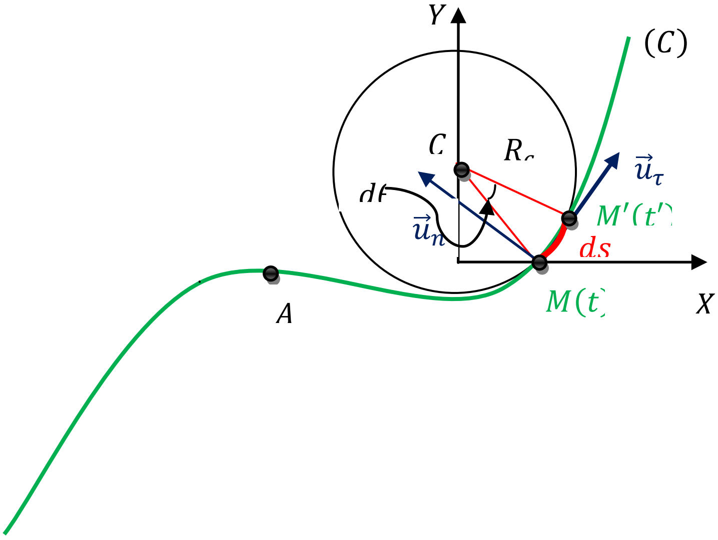
2.4.3 *Frenet frame (Intrinsic coordinates)*

In the case of planar motion, we can define at each point M on the trajectory the Frenet frame. For this purpose, at every point M, we define a vector $\vec{u}_\tau$ tangent to the trajectory and oriented in the direction of motion, and we define the $\vec{u}_n$ perpendicular to $\vec{u}_\tau$ and oriented towards the concavity of the trajectory. To complete the trihedron, we define a vector $\vec{B}$ such that the trihedron $(\vec{u}_n, \vec{u}_\tau, \vec{B})$ forms a right-handed system, i.e., $\vec{B} = \vec{u}_\tau \wedge \vec{u}_n$. The $(\vec{u}_n, \vec{u}_\tau, \vec{B})$ is called the *Serret-Frenet frame.*



2.4.3.1 Curvilinear abscissa:

In the case of curvilinear motion, it is sometimes useful to use the curvilinear abscissa to locate the position of the point particle. To do this, we fix a point A on the trajectory (see the figure). The curvilinear abscissa $s(t)$ is then defined as the curvilinear distance from the fixed point A to the point $M(t)$ occupied by the point particle at time t:



$$\widehat{AM} = arc(AM) = s(t)$$

At time $t' = t + dt$, with the point particle occupying the position $M'(t')$, we have the position vector:

$$\widehat{AM'} = arc(AM') = s(t') \quad ; \quad t' = t + dt$$

This vector represents the displacement of the point particle from the position $M(t)$ to the position $M'(t')$.

The infinitesimal displacement can be written as:

$$\widehat{MM'} = arc(MM') = s(t') - s(t) = ds$$

This represents the infinitesimal change in position from $M(t)$ to the position $M'(t')$.

ds is an arc of a circle with center C and radius R_C, known as the radius of curvature. The vectors $\vec{u}_n$ and $\vec{u}_\tau$ can then be obtained analytically as follows:

$$\boxed{\vec{u}_\tau = \frac{d\overrightarrow{OM}}{ds} \quad ; \quad \vec{u}_n = \frac{R_C\, d\vec{u}_\tau}{ds}}$$

2.4.3.2 Velocity vector in the Frenet frame:

By differentiating the position vector with respect to time, we obtain the expression for the velocity vector in the Frenet frame:

$$\boxed{\vec{V}(M/R) = \frac{ds}{dt}\vec{u}_\tau}$$

Indeed, we have already seen that $d\overrightarrow{OM} = \overrightarrow{MM'} = ds\vec{u}_\tau$, which gives us the expression for the velocity vector as follows:

$$\vec{V}(M/R) = \frac{d\overrightarrow{OM}}{dt} = \frac{ds}{dt}\vec{u}_\tau$$

The acceleration vector in the Frenet frame is given by:

$$\boxed{\vec{\gamma}(M/R) = \frac{dV}{dt}\vec{u}_\tau + \frac{V^2}{R_C}\vec{u}_n}$$

The acceleration vector can be decomposed into a tangential component, called the *tangential acceleration*:

$$\vec{\gamma}_\tau = \frac{dV}{dt}\vec{u}_\tau$$

and a normal component called the *normal acceleration*:

$$\vec{\gamma}_n = \frac{V^2}{R_C}\vec{u}_n$$

Where $$\vec{\gamma}(M/R) = \vec{\gamma}_\tau + \vec{\gamma}_n$$

Or in terms of magnitudes:

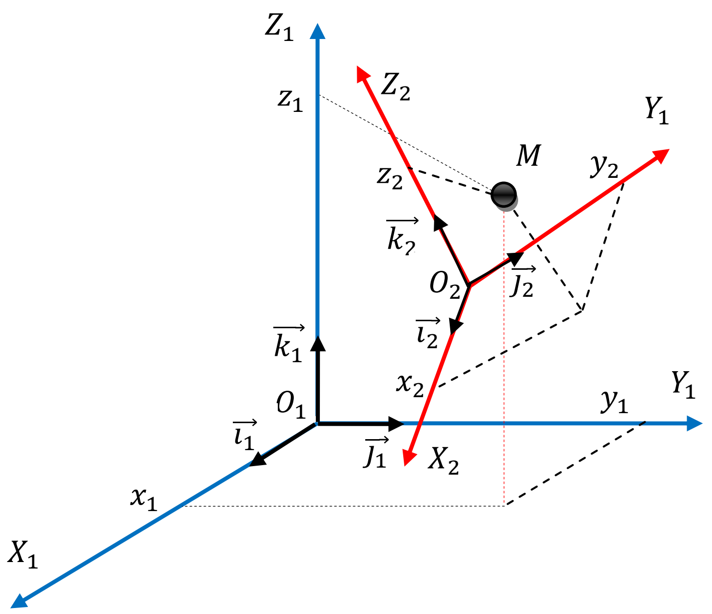
$$\gamma^2 = \gamma_\tau^2 + \gamma_n^2$$

We can observe that the magnitude of the normal acceleration component is always positive, indicating that the normal acceleration is always directed towards the concavity of the trajectory.

II.2 Kinematics with Change of Reference Frame

2.5 Relative Motion and Absolute Motion

We consider two reference frames, $R_1(O_1, X_1, Y_1, Z_1)$ and $R_2(O_2, X_2, Y_2, Z_2)$, ith respective bases $(\vec{\imath}_1, \vec{\jmath}_1, \vec{k}_1)$ et $(\vec{\imath}_2, \vec{\jmath}_2, \vec{k}_2)$, in motion relative to each other. We assume that R_1 is fixed and is called the absolute reference frame. The reference frame R_2 is then called the relative reference frame, as it is in motion with respect to R_1. We study the motion of a material point M with respect to both reference frames.



2.5.1 Absolute Motion of *M*-point

The motion of M with respect to the absolute reference frame is called absolute motion. The position of point M is determined by the Cartesian coordinates in reference frame R_1 (see figure above).

$$\overrightarrow{O_1M} = x_1\vec{\imath}_1 + y_1\vec{\jmath}_1 + z_1\vec{k}_1$$

The absolute velocity of M is the velocity of the material point M with respect to the absolute reference frame. It is obtained by taking the derivative with respect to time of the position vector in reference frame R_1:

$$\vec{V}(M/R_1) = \left.\frac{d\overrightarrow{O_1M}}{dt}\right|_{R_1}$$

The vectors of the base $(\vec{\imath}_1, \vec{\jmath}_1, \vec{k}_1)$, being related to reference frame R_1 have their respective temporal derivatives equal to zero: $\left.\frac{d\vec{\imath}_1}{dt}\right|_{R_1} = \left.\frac{d\vec{\jmath}_1}{dt}\right|_{R_1} = \left.\frac{d\vec{k}_1}{dt}\right|_{R_1} = \vec{0}$. Therefore, it is sufficient to differentiate the components:

$$\vec{V}(M/R_1) = \dot{x}_1\vec{\imath}_1 + \dot{y}_1\vec{\jmath}_1 + \dot{z}_1\vec{k}_1$$

The absolute acceleration is obtained by differentiating the absolute velocity with respect to time in the absolute reference frame:

$$\vec{\gamma}(M/R_1) = \left.\frac{d\vec{V}(M/R_1)}{dt}\right|_{R_1}$$

Here again, it is sufficient to differentiate the components of the absolute velocity vector:

$$\vec{\gamma}(M/R_1) = \ddot{x}_1\vec{\imath}_1 + \ddot{y}_1\vec{\jmath}_1 + \ddot{z}_1\vec{k}_1$$

2.5.2 Relative Motion of M

The motion of C with respect to the relative reference frame is called relative motion. The position of point M is determined by the Cartesian coordinates in reference frame R_2 (*see previous figure*).

$$\overrightarrow{O_2M} = x_2\vec{\imath}_2 + y_2\vec{\jmath}_2 + z_2\vec{k}_2$$

The relative velocity of M is the velocity of the point mass with respect to the relative reference frame. It is obtained by differentiating the position vector with respect to time in reference frame R_2:

$$\vec{V}(M/R_2) = \left.\frac{d\overrightarrow{O_2M}}{dt}\right|_{R_2}$$

In this case, the vectors of the base $(\vec{\imath}_2, \vec{\jmath}_2, \vec{k}_2)$, being related to reference frame R_2 have their respective time derivatives equal to zero: $\left.\frac{d\vec{\imath}_2}{dt}\right|_{R_2} = \left.\frac{d\vec{\jmath}_2}{dt}\right|_{R_2} = \left.\frac{d\vec{k}_2}{dt}\right|_{R_2} = \vec{0}$. once again, it is sufficient to differentiate the components:

$$\vec{V}(M/R_2) = \dot{x}_2\vec{\imath}_2 + \dot{y}_2\vec{\jmath}_2 + \dot{z}_2\vec{k}_2$$

The relative acceleration is obtained by differentiating the relative velocity with respect to time in the reference frame R_2:

$$\vec{\gamma}(M/R_2) = \left.\frac{d\vec{V}(M/R_2)}{dt}\right|_{R_1}$$

Its expression in the relative base is:

$$\vec{\gamma}(M/R_2) = \ddot{x}_2\vec{\imath}_2 + \ddot{y}_2\vec{\jmath}_2 + \ddot{z}_2\vec{k}_2$$

2.5.3 Special case: R_2 in rectilinear translation with respect to R_1

In this case, the vectors of the relative base $(\vec{\imath}_2, \vec{\jmath}_2, \vec{k}_2)$, are also fixed with respect to the reference frame R_1:

$$\left.\frac{d\vec{\imath}_2}{dt}\right|_{R_1} = \left.\frac{d\vec{\jmath}_2}{dt}\right|_{R_1} = \left.\frac{d\vec{k}_2}{dt}\right|_{R_1} = \vec{0}$$

2.5.4 Special case: R_2 in rotation with respect to R_1

If the reference frameR_2 is rotating with respect to reference frame R_1 with an angular velocity $\vec{\omega}(R_2/R_1)$. then the vectors of the relative base are also rotating with the same angular velocity $\vec{\omega}(R_2/R_1) = \vec{\omega}$. Using the result expressed in the remark at the end of the previous paragraph, we obtain the respective time derivatives of the base vectors:

$$\left.\frac{d\vec{\imath}_2}{dt}\right|_{R_1} = \vec{\omega} \wedge \vec{\imath}_2 \quad , \quad \left.\frac{d\vec{\jmath}_2}{dt}\right|_{R_1} = \vec{\omega} \wedge \vec{\jmath}_2 \quad , \quad \left.\frac{d\vec{k}_2}{dt}\right|_{R_1} = \vec{\omega} \wedge \vec{k}_2$$

2.5.5 General case: R_1 in arbitrary motion with respect to R_2

Any motion of one reference frame with respect to the other can be decomposed into a combination of a rectilinear translation and a rotation, highlighting the importance of these two types of motion.

2.6 Derivation in a moving frame

Throughout the following (unless otherwise specified), we will consider two reference frames R_1 and R_2 associated with the coordinate systems $(O_{1\,,}X_1,Y_1,Z_1)$ and$(O_{2\,,}X_2,Y_2,Z_2)$ respectively. These reference frames are characterized by the orthonormal bases $(\vec{\imath}_1,\vec{\jmath}_1,\vec{k}_1)$ et $(\vec{\imath}_2,\vec{\jmath}_2,\vec{k}_2)$. respectively. We assume that R_2 (arbitrary) motion relative to R_1 and this motion is characterized by the angular velocity $\vec{\omega} = \vec{\omega}(R_2/R_1)$.

Let $\vec{A}$ be a vector defined by its expression in the relative frame R_2:

$$\vec{A} = x_2\vec{\imath}_2 + y_2\vec{\jmath}_2 + z_2\vec{k}_2$$

To differentiate the vector $\vec{A}$ with respect to the reference frame R_1, we need to differentiate both its components and the vectors of the moving base $(\vec{\imath}_2,\vec{\jmath}_2,\vec{k}_2)$ with respect to R_1:

$$\left.\frac{d\vec{A}}{dt}\right|_{R_1} = \dot{x}_2\vec{\imath}_2 + x_2\frac{d\vec{\imath}_2}{dt} + \dot{y}_2\vec{\jmath}_2 + y_2\frac{d\vec{\jmath}_2}{dt} + \dot{z}_2\vec{k}_2 + z_2\frac{d\vec{k}_2}{dt}$$

We have seen that the derivative of a unit vector $\vec{u}$ undergoing rotation with an angular velocity $\vec{\omega}$ with respect to a fixed frame is given by $\frac{d\vec{u}}{dt} = \vec{\omega} \wedge \vec{u}$. By replacing $\vec{u}$ with the vectors of the base $(\vec{\imath}_2,\vec{\jmath}_2,\vec{k}_2)$, we obtain:

$$\left.\frac{d\vec{A}}{dt}\right|_{R_1} = \dot{x}_2\vec{\imath}_2 + +\dot{y}_2\vec{\jmath}_2 + \dot{z}_2\vec{k}_2 + x_2(\vec{\omega} \wedge \vec{\imath}_2) + y_2(\vec{\omega} \wedge \vec{\jmath}_2) + z_2(\vec{\omega} \wedge \vec{k}_2)$$

Indeed, we have $\dot{x}_2\vec{\imath}_2 + +\dot{y}_2\vec{\jmath}_2 + \dot{z}_2\vec{k}_2 = \left.\frac{d\vec{A}}{dt}\right|_{R_2}$ which represents the derivative of vector $\vec{A}$ in the relative reference frame and $x_2(\vec{\omega} \wedge \vec{\imath}_2) + y_2(\vec{\omega} \wedge \vec{\jmath}_2) + z_2(\vec{\omega} \wedge \vec{k}_2) = \vec{\omega} \wedge (x_2\vec{\imath}_2 + y_2\vec{\jmath}_2 + z_2\vec{k}_2) = \vec{\omega} \wedge \vec{A}$

This equation shows that the cross product of the angular velocity $\vec{\omega}$ with the vector $\vec{A}$ yields the same result as the derivative of $\vec{A}$ in the reference frame R_2:

$$\left.\frac{d\vec{A}}{dt}\right|_{R_1} = \left.\frac{d\vec{A}}{dt}\right|_{R_2} = \vec{\omega}(R_2/R_1) \wedge \vec{A}$$

2.7 Velocity composition

The law of velocity composition is written as follows:

$$\vec{V}_a = \vec{V}_r + \vec{V}_e$$

where:

$\vec{V}_a = \vec{V}(M/R_1) = \left.\frac{d\overrightarrow{O_1M}}{dt}\right|_{R_1}$: *is the absolute velocity of the material point,*

$\vec{V}_r = \vec{V}(M/R_2) = \left.\frac{d\overrightarrow{O_2M}}{dt}\right|_{R_2}$: *is the relative velocity of the material point,*

$\vec{V}_e = \left.\frac{d\overrightarrow{O_1O_2}}{dt}\right|_{R_1} + \vec{\omega}(R_2/R_1) \wedge \overrightarrow{O_2M}$: *is the entrainment velocity.*

2.8 Composition of Accelerations

The law of decomposition of accelerations is written as follows:

$$\vec{\gamma}_a = \vec{\gamma}_r + \vec{\gamma}_e + \vec{\gamma}_c$$

Where:

$$\vec{\gamma}_a = \vec{\gamma}(M/R_1) = \left.\frac{d\vec{V}_a}{dt}\right|_{R_1}$$

is the absolute acceleration of the material point,

$$\vec{\gamma}_r = \vec{\gamma}(M/R_2) = \left.\frac{d\vec{V}_r}{dt}\right|_{R_1} = \left.\frac{d^2\overrightarrow{O_2M}}{dt^2}\right|_{R_2}$$

is the relative acceleration of the material point,

$$\vec{\gamma}_e = \left.\frac{d^2\overrightarrow{O_1O_2}}{dt^2}\right|_{R_1} + \frac{d\vec{\omega}(R_2/R_1)}{dt} \wedge \overrightarrow{O_2M} + \vec{\omega}(R_2/R_1) \wedge \left(\vec{\omega}(R_2/R_1) \wedge \overrightarrow{O_2M}\right)$$

The entrainment acceleration, denoted as $\vec{\gamma}_e$, is the acceleration of entrainment.

$$\vec{\gamma}_c = 2\vec{\omega}(R_2/R_1) \wedge \vec{V}_r$$

is the complementary acceleration, also known as the Coriolis acceleration.

Exercises

Kinematics

II.1 Kinematics

Exercise 1.1

The rectilinear motion of a point is defined by the following equation:

$$S = 2t^3 - 9t^2 + 12t + 1$$

1. Calculate the velocity and acceleration at time t.
2. Study the motion of the point as t increases from 0 to ∞ (Indicate the direction of motion and whether it is accelerating or decelerating).

Exercise 1.2

Determine the trajectory of the planar motion defined by the equations:

$$x = \sin^2 t \quad ; \quad y = 1 + \cos 2t$$

Plot this trajectory in the Oxy coordinate system.

Exercise 1.3

In an orthonormal coordinate system $(O, \vec{i}, \vec{j}, \vec{k})$, the motion of a mobile M is defined by the following equations:

$x = t^3 - 3t \quad ; \quad y = -3t^2 \quad ; \quad z = t^3 - 3t$

1. Calculate the coordinates at time t of the velocity vector $\vec{v}(t)$ and the acceleration vector $\vec{a}(t)$ of the mobile *M*.
2. Calculate the magnitude of the velocity vector $\vec{v}(t)$ and show that this vector makes a constant angle with *Oz*.

Exercise 1.4

A material point is moving in a plane *R(O, x, y)* starting from time *t=1*. Its equations of motion are:

$$x = \ln t \quad ; \quad y = t + \frac{1}{t}$$

1. Write the equation of the trajectory.
2. Calculate the algebraic values of the velocity and acceleration at time *t*.

Exercise 1.5

The equations of motion for a point M in a reference frame $R(O,\vec{\imath},\vec{\jmath},\vec{k})$ are given by:

$x = \frac{1}{2}bt^2$; $y = ct$; $z = \frac{3}{2}bt^2$

Where b and c are positive constants.

1. Find the velocity and acceleration vectors as well as their magnitudes.
2. What is the equation of the trajectory for the point m, which represents the vertical projection of the moving point M onto the XOY plane?

Exercise 1.6

Consider the trajectory defined by:

$$\vec{r} = 3\cos 2t\,\vec{\imath} + 3\sin 2t\,\vec{\jmath} + (8t-4)\,\vec{k}$$

1. Find the unit vector $\vec{T}$ tangent to the trajectory.
2. If $\vec{v}$ is the position vector of a point moving on C at time t, verify that in this case $\vec{v} = v.\vec{T}$.

Exercise 1.7

A point M describes a circular helix with the Z-axis as its axis. The parametric equations for the helix are:

$x = R\cos\theta$; $y = R\sin\theta$; $z = h\theta$

where R is the radius of the cylindrical surface on which the helix is traced, h is a constant, and θ is the angle that the projection OM' of OM onto the XOY plane makes with the OX axis.

1. Provide the expressions for velocity and acceleration in cylindrical coordinates.
2. Show that the velocity vector forms a constant angle with the XOY plane.
3. Demonstrate that the rotational motion is uniform, that the acceleration vector passes through the axis of the cylinder, and is parallel to the XOY plane. Calculate the radius of curvature.

Exercise 1.8

A particle moves in space according to the following equations:

$x = R\cos\omega t$; $y = R\sin\omega t$; $z = \alpha t$

Where R, ω, and α are positive constants. Let m be the projection of M onto the XY plane:

1. What is the nature of the trajectory of m in the XY plane?
2. What is the nature of the motion of m along the OZ axis?
3. Deduce the nature of the trajectory of the particle M. In the cylindrical coordinate system:
4. Write the expression of the position vector $\overrightarrow{OM}$ and represent the basis $(\vec{u}_\rho, \vec{u}_\varphi, \vec{u}_z)$ at a point M in space.
5. Find the velocity and acceleration of M, as well as their magnitudes. Determine their directions and represent them at a point in space.
6. Deduce the radius of curvature.

Solution

Exercise 1.1

1- To calculate the velocity, we simply differentiate the equation of motion with respect to time:

$$v = \frac{dS}{dt} = 6t^2 - 18t + 12$$

By differentiating the velocity with respect to time, we obtain the acceleration:

$$a = \frac{dv}{dt} = 12t - 18$$

2- Studying the motion of the object requires a mathematical analysis of the function $S = 2t^3 - 9t^2 + 12t + 1$.

The motion is accelerated or decelerated depending on the sign of the product av. The direction of motion is indicated by the sign of v. Let's construct the table of variations:

$v = 6t^2 - 18t + 12 = 0 \qquad \Longrightarrow \; t = 0\,; \quad t = 2$

$a = 12t - 18 = 0 \qquad \Longrightarrow \; t = 1.5$

t	*0*	*1*	*1.5*	*2* *∞*
v	+ 0	-	- 0	+
a	-	- 0	+	+
ay	-	+	-	+
Mvt	*Decelerating (+)*	*Accelerating (-)*	*Decelerating (+)*	*Accelerating (+)*

Exercise 1.2

Let's start with the trigonometric transformation:

$$\cos 2t = 2\cos^2 t - 1\,.$$

Replacing it in the expression of y, we have: $y = \cos^2 t$

Another trigonometric transformation leads us to:

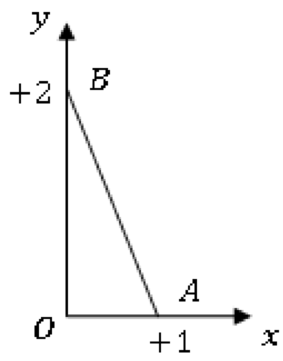
$$y = 2(\sin^2 t - 1)$$

We just need to replace $\sin^2 t$ with x to obtain the equation of the trajectory, which is:

$y = 2(1 - x)$

To draw the trajectory, we notice that $0 \le x \le +1$, because for any t, $0 \le \sin^2 t = x \le +1$.

We deduce that the trajectory is a straight line segment connecting points A(+1,0) and B(0,+2).



Exercise 1.3

1- Two consecutive differentiations of the equations of motion lead us to the expressions of the coordinates of the velocity and acceleration vectors of the mobile at time *t*:

$$\vec{v} = \begin{cases} \dot{x} = v_x = 3(t^2 - 1) \\ \dot{y} = v_y = -6t \\ \dot{z} = v_z = 3(t^2 + 1) \end{cases} \; ; \; \vec{a} = \begin{cases} \ddot{x} = a_x = 6t \\ \ddot{y} = a_y = -6 \\ \ddot{z} = a_z = 6t \end{cases}$$

2- The magnitude of the velocity vector is given by $v^2 = 18(t^2 + 1)^2 \implies v = 3\sqrt{2}(t^2 + 1)$. Now let's calculate the angle between the vector $\vec{v}$ and Oz. To do this, we need to calculate the magnitude of the dot product:

$$\vec{v}.\vec{k} = v.k\cos(\vec{v},\vec{k}) \implies \cos(\vec{v},\vec{k}) = \frac{\vec{v}.\vec{k}}{v}$$

$$\vec{v} = \begin{cases} \dot{x} \\ \dot{y} \\ \dot{z} \end{cases} \; ; \; \vec{k} = \begin{cases} 0 \\ 0 \\ 1 \end{cases}$$

$$\vec{v}.\vec{k} = (\dot{x}.0) + (\dot{y}.0) + (\dot{z}.0) = 3(1 + t^2)$$

$$\cos(\vec{v},\vec{k}) = \frac{\vec{v}.\vec{k}}{v} = \frac{3(1+t^2)}{3\sqrt{2}(t^2+1)} \implies \cos(\vec{v},\vec{k}) = \frac{\sqrt{2}}{2} \implies (\vec{v}, Oz) = \frac{\pi}{4} rad$$

Exercise 1.4

1- Let's eliminate time between the two equations of motion to obtain the equation of the trajectory:

$$x = \ln t \implies t = e^x$$

$$y = e^x + \frac{1}{e^x} \implies \boxed{y = e^x + e^{-x}}$$

2- Let's calculate the magnitudes of velocity and acceleration at time t by successively differentiating the two equations of motion with respect to time:

$$\left. \begin{aligned} v_x &= \frac{1}{t} \\ v_y &= 1 - \frac{1}{t^2} \end{aligned} \right| \implies v = \sqrt{\left(\frac{1}{t}\right)^2 + \left(1 - \frac{1}{t^2}\right)^2} \; ; \; \boxed{v = \sqrt{\frac{1}{t^4} - \frac{1}{t^2} + 1}}$$

$$\left. \begin{aligned} a_x &= -\frac{1}{t^2} \\ a_y &= -\frac{2t}{t^4} \end{aligned} \right| \implies a = \sqrt{\left(\frac{1}{t^2}\right)^2 + \left(\frac{2}{t^3}\right)^2} \; ; \; a = \sqrt{\frac{1}{t^4} - \frac{1}{t^2} + 1}$$

Exercice 1.5

1- The velocity vector:

Let's write the expression for the position vector:

$$\vec{r} = \frac{1}{2}bt^2\vec{\imath} + ct\vec{\jmath} + \frac{3}{2}bt^2\vec{k}$$

2- To obtain the velocity vector, we need to differentiate the position vector with respect to time:

$$\dot{x} = v_x = bt \quad , \quad \dot{y} = v_y = c \quad \dot{z} = v_z = 3bt$$

$$\boxed{\vec{v} = bt\vec{\imath} + c\,\vec{\jmath} + 3bt\,\vec{k}} \quad \Longrightarrow \quad \boxed{v = \sqrt{10(bt^2) + c^2}}$$

Let's differentiate the velocity vector to obtain the acceleration vector:

Differentiating each component of the velocity vector with respect to time, we get:

$$\ddot{x} = a_x = b \quad , \quad \ddot{y} = a_y = 0 \quad \ddot{z} = a_z = 3b$$

$$\boxed{\vec{a} = b\vec{\imath} + 3b\,\vec{k}} \quad ; \quad \boxed{a = 2b}$$

3- To eliminate time between the two equations of motion, *x(t)* and *y(t)*, and obtain the equation of the trajectory of point *m*, we can equate the expressions for x and y and solve for one variable in terms of the other.

$$x = \frac{1}{2}bt^2 \quad \Longrightarrow \quad t^2 = \frac{2x}{b} \quad ; \quad y = c\sqrt{\frac{2x}{b}}$$

This is a quadratic equation in terms of t. Solving it will give us the values of t where the trajectory intersects the x-y plane. From these values, we can obtain the corresponding values of x and y to describe the equation of the trajectory.

Exercice 1.6

1- The tangent vector to the trajectory is the velocity vector $\vec{v} = \frac{d\vec{r}}{dt}$. In Cartesian coordinates, the velocity vector is given by:

$$\vec{v} = \frac{d\vec{r}}{dt} = -6\sin 2t\,\vec{\imath} + 6\cos 2t\,\vec{\jmath} + 8\,\vec{k}$$

Its magnitude is equal to : $v = \sqrt{36 + 64} \quad \Longrightarrow \boxed{v = 10m.s^{-2}}$

The unit vector $\vec{T}$ tangent to the trajectory C is parallel to the velocity vector $\vec{v}$. To obtain the unit vector $\vec{T}$, we divide the velocity vector $\vec{v}$ by its magnitude:

$$\vec{T} = \frac{\vec{v}}{v} = -\frac{3}{5}\sin 2t\,\vec{\imath} + \frac{3}{5}\cos 2t\,\vec{\jmath} + \frac{4}{5}\vec{k}$$

2- If $\vec{v}$ is the position vector of point M at time t, then $\vec{v} = \frac{d\vec{r}}{dt}$ represents the derivative of the position vector with respect to time.

$$\vec{v} = \frac{d\vec{r}}{dt} = -6\sin 2t\,\vec{\imath} + 6\cos 2t\,\vec{\jmath} + 8\vec{k} \quad \Longrightarrow \quad \vec{v} = -6\sin 2t\,\vec{\imath} + 6\cos 2t\,\vec{\jmath} + 8\vec{k}$$

$$\vec{v} = 10\left(-\frac{3}{5}\sin 2t\,\vec{\imath} + \frac{3}{5}\cos 2t\,\vec{\jmath} + \frac{4}{5}\vec{k}\right)$$

$$\vec{v} = 10\vec{u}_T = 10\vec{T} \quad \Longrightarrow \quad \vec{v} = v.\vec{T}$$

Exercise 1.7

1- We know that the position vector in cylindrical coordinates is expressed as:

$$\overrightarrow{OM} = \vec{r} = \rho.\vec{u}_\rho + z\vec{u}_z$$

From the given information, we can deduce the velocity vector by differentiation:

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \dot{\vec{r}} = \dot{\rho}.\vec{u}_\rho + \rho.\dot{\vec{u}}_\rho + \dot{z}\vec{u}_z$$

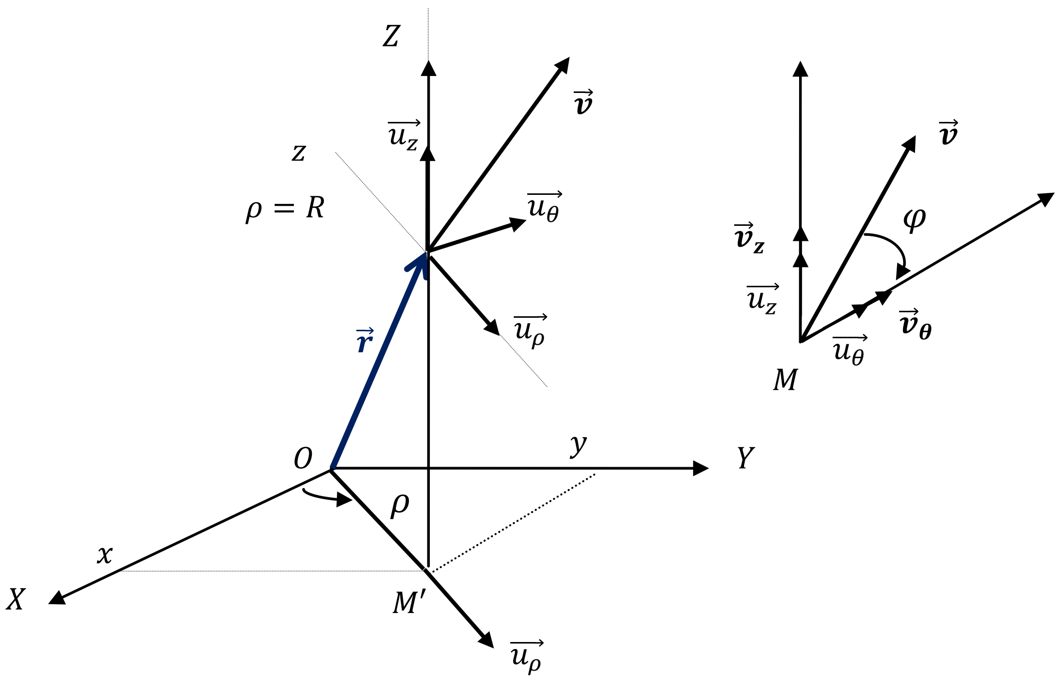
$$\left.\begin{array}{r} \rho = R \Longrightarrow \dot{\rho} = 0 \\ \dot{\vec{u}}_\rho = \dot{\theta}.\vec{u}_\theta \\ \dot{z} = h.\dot{\theta} \end{array}\right\} \Longrightarrow \vec{v} = R\dot{\theta}.\vec{u}_\theta + h.\dot{\theta}\vec{u}_z$$

Similarly, the acceleration vector $\vec{a}$ is given by $\vec{a} = d\vec{v}/dt$.

$$\vec{a} = \frac{d^2\overrightarrow{OM}}{dt^2} = \dot{\vec{v}} = R\ddot{\theta}\vec{u}_\theta + R\dot{\theta}.\dot{\vec{u}}_\theta + h.\ddot{\theta}\vec{u}_z$$

$$\left.\begin{array}{r} \ddot{\theta} = 0 \\ \dot{\vec{u}}_\theta = -\dot{\theta}\vec{u}_\rho \end{array}\right\} \Longrightarrow R\ddot{\theta}\vec{u}_\theta + R\dot{\theta}.\dot{\vec{u}}_\theta + h.\ddot{\theta}\vec{u}_z$$

2- The unit vector $\vec{u}_\theta$ is parallel to the XY plane, so the angle formed by the velocity vector with the XY plane is equal to the angle formed by the unit vector $\vec{u}_\theta$ with the XY plane. Additionally, $\vec{u}_\theta$ is perpendicular to $\vec{u}_z$ ($\vec{u}_\theta \perp \vec{u}_z$).



$$\tan(\vec{v}, \vec{u}_\theta) = \frac{v_z}{v_\theta} = \frac{h.\dot{\theta}}{R.\dot{\theta}} \quad \Longrightarrow \quad \boxed{\tan(\vec{v}, \vec{u}_\theta) = \frac{h}{R} = Cte}$$

3- The motion is uniform rotational motion, which means that $\theta = \omega = Ct$. In this case, the acceleration vector $\vec{a}$ is given by $\vec{a} = -R\omega^2\vec{u}_\rho$.

The acceleration vector $\vec{a}$ is parallel to $\vec{u}_\rho$, which means it is centripetal and confirms that it passes through the axis of the cylinder. $\vec{u}_\rho$ belongs to the OXY plane, and since $\vec{a}$ is parallel to $\vec{u}_\rho$, it shows that the acceleration is parallel to the OXY plane.

We have just demonstrated that the acceleration is centripetal, which implies that:

$$\left.\begin{array}{c} r = \dfrac{v^2}{a_x} = \dfrac{v^2}{a} \\ v^2 = R^2\omega^2 + h^2\omega^2 \\ a = R^2\omega^2 \end{array}\right\} \quad \Longrightarrow \vec{r} = \frac{R^2\omega^2 + h^2\omega^2}{R\omega^2} \; ; \; r = \frac{R^2 + h^2}{R}$$

This confirms that the acceleration is directed towards the axis of the cylinder.

Exercise 1.8

1- a) The motion of point m takes place in the *XOY* plane. To obtain the equation of the trajectory for this point, we eliminate time between the equations $x(t)$ and $y(t)$. We obtain $x^2 + y^2 = R^2$, which is the equation of a circle with center $(0,0)$ and radius *R*.

b) Along the *OZ* axis, the equation of the trajectory $z = \alpha t$ indicates that the motion is vertically uniform rectilinear.

c) The trajectory of the mobile is the combination of the planar motion and the vertical motion, resulting in a helical motion.

2) In the cylindrical coordinate system:

a) The position vector is given by:

$$\overrightarrow{OM} = \vec{r} = \rho.\vec{u}_\rho + z.\vec{u}_z \quad \Leftrightarrow \quad \overrightarrow{OM} = \vec{r} = R.\vec{u}_\rho + z.\vec{u}_z$$

b) The velocity and acceleration of point *M* are:

$$\left.\begin{aligned} \vec{v} = \dot{\vec{r}} &= \dot{\rho}.\vec{u}_\rho + \rho.\dot{\vec{u}}_\rho + z.\vec{u}_z \\ \dot{\vec{u}}_\rho &= \dot{\varphi}.\vec{u}_\varphi = +R\omega.\vec{u}_\varphi \end{aligned}\right\} \Longrightarrow \boxed{\vec{v} = R\omega.\vec{u}_\varphi + b.\vec{u}_z} \quad , \quad \boxed{v = \sqrt{R^2\omega^2 + b^2}}$$

$$\left.\begin{aligned} \vec{a} &= R\omega.\dot{\vec{u}}_\varphi \\ \dot{\vec{u}}_\varphi &= -\dot{\varphi}.\vec{u}_\rho = -R\omega.\vec{u}_\rho \end{aligned}\right\} \Longrightarrow \boxed{\vec{a} = -R\omega^2.\vec{u}_\rho} \quad , \quad \boxed{v = R\omega^2}$$

The angle between the velocity vector and the $\vec{u}_\varphi$ vector is:

$$\boxed{\tan\beta = \frac{v_z}{v_\varphi} = \frac{b}{R\omega}}$$

The acceleration is centripetal, meaning it is directed towards the center of the circular trajectory.

c) The radius of curvature is given by:

$$\left.\begin{aligned} r &= \frac{v^2}{a_N} \\ a_N{}^2 = a^2 - a_T{}^2 &\Longrightarrow a_N{}^2 = R^2\omega^4 - (R^2\omega^2 + b^2) \\ a_T{}^2 &= R^2\omega^2 + b^2 \end{aligned}\right\}$$

$$\Rightarrow \boxed{r = \frac{R^2\omega^2 + b^2}{\sqrt{R^2\omega^2(\omega^2 - 1) - b^2}}}$$

II.2 Kinematics with Change of Reference Frame

Exercise 1.1

While driving in the rain at 100 km/h on a flat road, a driver notices that raindrops, seen through the side windows of his car, have trajectories that make an angle of 80° with the vertical. After stopping his car, he notices that the rain is actually falling vertically.

Calculate the speed of the rain relative to the stationary car and relative to the car moving at 100 km/h.

Exercise 1.2

In the fixed coordinate system *OXY*, consider a system of two mobile axes *Oxy*, where the *Ox* axis forms an angle θ with the *OX* axis. A material point *M* moves along the *Ox* axis, and its position is defined by $r = OM$.

Calculate:

1. The relative velocity and acceleration of the point,
2. The driving velocity and acceleration,
3. The Coriolis acceleration,
4. Determine the velocity and acceleration of point *M* in polar coordinates.

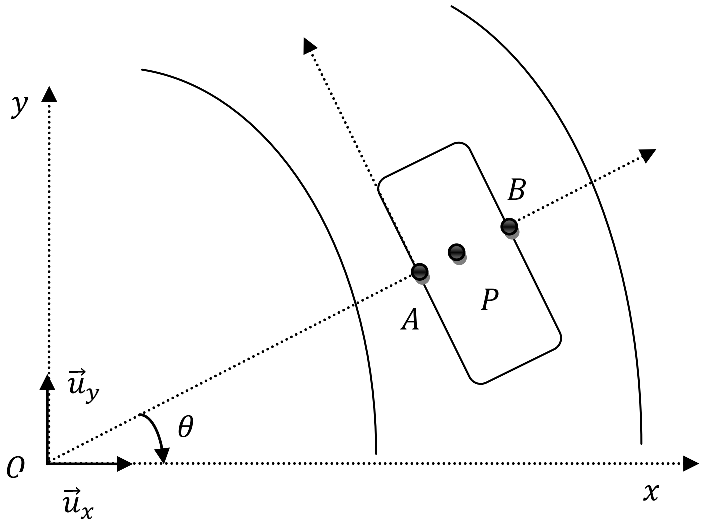
Exercise 1.3

At a specific moment taken as the origin of time $(t = 0)$, a bus takes a turn with a constant angular velocity ω_0. O is the center of the turn, and the distance $OA = R$. At that moment, a passenger *P*, stationary at *A*, rushes directly towards an available seat at *B*, with a constant acceleration a_0 (see figure).

1- Analysis in the bus reference frame R_A. Specify the chosen coordinate system and the nature of *P*s motion. Determine, in terms of the given data and *t*, the acceleration vector $\vec{a}_r$ and the velocity vector $\vec{v}_r$ of point *P*, as well as the equation of motion.

2- Analysis in the Earth reference frame R_T. Specify the chosen coordinate system. Determine the velocity vector $\vec{v}_T$ and the acceleration vector $\vec{a}_T$ of point P. Give the equation of the trajectory of point P in polar coordinates ($\rho = OP$ as a function of θ).

3- Using the laws of velocity and acceleration composition, derive the vectors $\vec{v}_T$ and $\vec{a}_T$ from the vectors $\vec{v}_r$ and $\vec{a}_r$. Clearly indicate the different terms involved in these laws, specifying their meaning and expression.



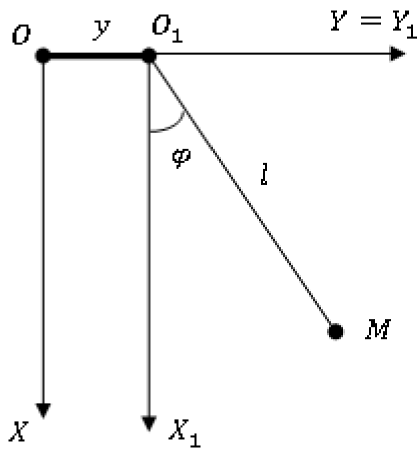
4- Numerical applications: $a_0 = 6\,m.s^{-2}$, $\omega_0 = 1/6\ rad.s^{-1}$, $R = 120\,nm$ and $AB = 3\,m$.

At what velocity does P reach the seat B and in how much time? How far has the bus traveled, and by what angle has it turned?

Exercise 1.4

Consider a material point M suspended from an inextensible string of length l. The suspension point O_1 of the pendulum, formed by the point M and the string, is in motion in the reference frame $R(O, X, Y, Z)$ along the OY axis. The position of O_1 is determined by y. The motion of M takes place in the OXY plane, as shown in the figure. Let $R_1(O_1, X_1, Y_1, Z_1)$ be the reference frame with origin O_1, and its axes remain constantly parallel to those of R.

1. Calculate the velocity and acceleration of M in R_1.
2. Calculate the entrainment velocity, entrainment acceleration, and Coriolis acceleration experienced by M due to the motion of R_1 relative to R.
3. Deduce the velocity and acceleration of M in R.



Solution

Exercise 1.1

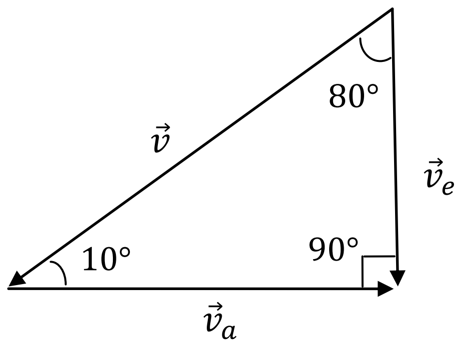
Let $\vec{v}_a$ be the rainfall velocity relative to the ground, $\vec{v}$ be the rainfall velocity relative to the vehicle, and $\vec{v}_e$be the velocity of the vehicle relative to the ground. Let's represent the three vectors and apply the law of sines:

The rainfall velocity relative to the stationary vehicle is:

$$\frac{v_a}{\sin 10°} = \frac{v_r}{\sin 90°} \Longrightarrow \boxed{v_a = \frac{\sin 10°}{\sin 90°} v_r} \quad AN: v_a = 17.4\ km.h^{-1}$$

The rainfall velocity relative to the moving vehicle is:

$$\frac{v_r}{\sin 90°} = \frac{v_e}{\sin 80°} \Longrightarrow \boxed{v_r = \frac{\sin 90°}{\sin 80°} v_e} \quad AN: v_r = 117\ km.h^{-1}$$



Exercise 1.2

We are studying the motion of M in the $(\vec{u}_r, \vec{u}_\theta, \vec{u}_z)$ basis. The unit vectors are time-independent. Refer to the figure.

1. The position vector $\boxed{\overrightarrow{OM} = \vec{r} = \vec{r}'}$, the relative velocity vector: $\boxed{\vec{v}_r = \dot{r}.\vec{u}_r}$, and the relative acceleration vector: $\boxed{\vec{a}_r = \ddot{r}.\vec{u}_r}$.

2. The entrainment velocity, i.e., the velocity of the two mobile axes Oxy relative to the fixed plane OXY, is:

$$\left.\begin{array}{c}\vec{V}_e = \dfrac{d\overrightarrow{OO'}}{dt} + \vec{\omega} \wedge \overrightarrow{O'M} \\ \dfrac{d\overrightarrow{OO'}}{dt} = 0\ (O \equiv O') \\ \vec{\omega} = \dot{\theta}\vec{k} = \dot{\theta}.\vec{u}_z\end{array}\right| \Rightarrow \vec{V}_e = \vec{\omega} \wedge \overrightarrow{O'M} = \begin{vmatrix}\vec{u}_r & \vec{u}_\theta & \vec{u}_z \\ 0 & 0 & \dot{\theta} \\ r & 0 & 0\end{vmatrix} \Rightarrow \boxed{\vec{V}_e = r\dot{\theta}.\vec{u}_\theta}$$

To determine the entrainment acceleration, we differentiate the entrainment velocity $\vec{V}_e$ with respect to time:

$$\vec{a}_e = \frac{d^2\overrightarrow{OO'}}{dt^2} + \vec{\omega} \wedge \frac{d\overrightarrow{O'M}}{dt} + \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O'M}\ , \qquad \frac{d\overrightarrow{O'M}}{dt} = \vec{\omega} \wedge \overrightarrow{O'M}$$

$$\vec{a}_e = \frac{d^2\overrightarrow{OO'}}{dt^2} + \vec{\omega} \wedge \left(\vec{\omega} \wedge \overrightarrow{O'M}\right) + \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O'M}$$

$$\left.\begin{array}{l}\vec{\omega} \wedge \vec{\omega} \wedge \overrightarrow{O'M} = \dot{\theta}.\vec{u}_z \wedge r\dot{\theta}.\vec{u}_\theta = -r\dot{\theta}^2\vec{u}_r \\ \dfrac{d\vec{\omega}}{dt} \wedge \overrightarrow{O'M} = \begin{vmatrix}\vec{u}_r & -\vec{u}_\theta & \vec{u}_z \\ 0 & 0 & \ddot{\theta} \\ r & 0 & 0\end{vmatrix} = r\ddot{\theta}\vec{u}_\theta\end{array}\right| \Rightarrow \boxed{\vec{a}_e = -r\dot{\theta}^2\vec{u}_r + r\ddot{\theta}\vec{u}_\theta}$$

3. The Coriolis acceleration can be calculated as:

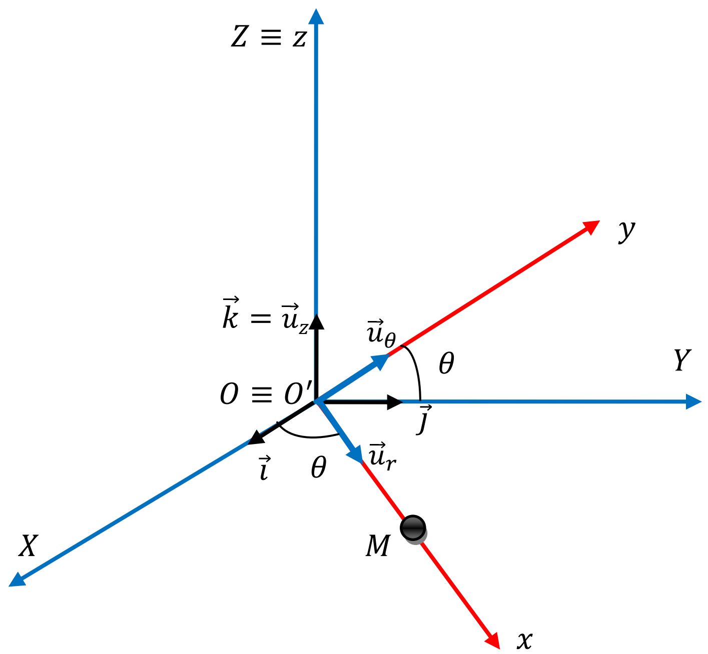
$$\vec{a}_c = 2\vec{\omega} \wedge \vec{v}_r = 2\begin{vmatrix}\vec{u}_r & -\vec{u}_\theta & \vec{u}_z \\ 0 & 0 & \dot{\theta} \\ r & 0 & 0\end{vmatrix} \Rightarrow \boxed{\vec{a}_c = 2\dot{r}\dot{\theta}\vec{u}_\theta}$$

The absolute velocity, which is the velocity of M relative to the fixed plane OXY, is given by:

$$\vec{v}_a = \vec{v}_e + \vec{v}_r \quad \Rightarrow \boxed{\vec{v}_a = \dot{r}.\vec{u}_r\ + r\theta\vec{u}_\theta}$$

And the absolute acceleration, which is the acceleration of M relative to the fixed plane OXY, is:

$$\vec{a}_a = \vec{a}_r + \vec{a}_e + \vec{a}_c \quad \Rightarrow \quad \vec{a}_a = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{u}_r\ + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{u}_\theta$$



Exercise 1.3

1- Consider reference frame R attached to the bus, with the basis $(A, \vec{u}_\rho, \ \vec{u}_\theta)$ fixed in the bus. Point P undergoes uniformly accelerated rectilinear motion with acceleration a_0 in the direction of $A \ \to B$ (direction of $\vec{u}_\rho$).

At $t = 0$, P is at position A.

$$a_r = a_0 \ \Rightarrow v_r = \ a_0 t \quad \Longrightarrow \quad AP = x = \frac{1}{2} a_0 t^2.$$

2- In the Earth reference frame R_T, with the basis $(A, \vec{u}_\rho, \ \vec{u}_\theta)$ in polar coordinates.

$$\overrightarrow{OP} = r\vec{u}_\rho = (R + x)\vec{u}_\rho \quad \Longrightarrow \quad \vec{v}_r = \dot{x}\vec{u}_\rho + (R + x)\dot{\theta}\vec{u}_\theta$$

$$= a_0 t\vec{u}_\rho + \left(R + \frac{1}{2} a_0 t^2\right) \omega_0 \vec{u}_\theta.$$

$$a_T = \ddot{x}\vec{u}_\rho + \dot{x}\dot{\theta}\vec{u}_\theta + \ \dot{x}\dot{\theta}\vec{u}_\theta - (R + x)\dot{\theta}^2\vec{u}_\rho$$

$$= \left(a_0 - \left(R + \frac{1}{2} a_0 t^2\right) \omega_0^2 \ \right)\vec{u}_\rho + 2a_0 t\omega_0 \vec{u}_\theta.$$

The trajectory equation is :

$$r = R + \frac{1}{2}a_0 t^2 \text{ et } \theta = \omega_0 t \implies r = R + \frac{1}{2}\frac{a_0}{\omega_0^2}\theta^2 \; (equation\ of\ spiral).$$

3) The trajectory equation is:

The reference frame R is rotating with angular velocity vector $\vec{\omega}(R/R_T) = \omega_0 \vec{u}_z$.

$$\vec{v}_T = \vec{v}_r + \vec{v}_e \text{ with } \vec{v}_r = a_0 t \vec{u}_\rho \text{ et}$$

$$\vec{v}_e = \frac{d\overrightarrow{OA}}{dt} + \left(\vec{\omega}(R/R_T) \wedge \overrightarrow{AP}\right) = \left(\vec{\omega}(R/R_T) \wedge \overrightarrow{OA}\right) + \left(\vec{\omega}(R/R_T) \wedge \overrightarrow{AP}\right)$$

$$\vec{v}_e = \left(\vec{\omega}(R/R_T) \wedge \overrightarrow{OP}\right) = (R + x)\omega_0 \vec{u}_\theta = \left(R + \tfrac{1}{2}a_0 t^2\right)\omega_0 \vec{u}_\theta.$$

The entrainment velocity corresponds to the velocity of point P relative to R_T if it were fixed in the bus at that moment. It would then have a uniform circular motion with radius $(R + x)$ and angular velocity ω_0, hence the expression of the velocity vector. Therefore:

$$\vec{v}_T = \vec{v}_r + \vec{v}_e \implies \vec{v}_T = a_0 t \vec{u}_\rho + \left(R + \frac{1}{2}a_0 t^2\right)\omega_0 \vec{u}_\theta$$

Law of acceleration composition:

$$\vec{a}_T = \vec{a}_r + \vec{a}_e + \vec{a}_c \text{ avec } \vec{a}_r = a_0 \vec{u}_\rho \text{ et}$$

$$\vec{a}_c = 2\vec{\omega}(R/R_T) \wedge \vec{v}_r = 2\omega_0 \vec{u}_z \wedge a_0 t \vec{u}_\rho = 2a_0 t \omega_0 \vec{u}_\theta.$$

$$\vec{a}_e = \frac{d^2\overrightarrow{OA}}{dt^2} + \frac{d\vec{\omega}(R/R_T)}{dt} \wedge \overrightarrow{AP} + \vec{\omega}(R/R_T) \wedge \left(\vec{\omega}(R/R_T) \wedge \overrightarrow{AP}\right)$$

$$= \vec{\omega}(R/R_T) \wedge \left(\vec{\omega}(R/R_T) \wedge \overrightarrow{OA}\right) + \vec{\omega}(R/R_T) \wedge \left(\vec{\omega}(R/R_T) \wedge \overrightarrow{AP}\right).$$

$$\vec{a}_e = \vec{\omega}(R/R_T) \wedge \left(\vec{\omega}(R/R_T) \wedge \overrightarrow{OP}\right) = \omega_0 \vec{u}_z \wedge \left(\omega_0 \vec{u}_z \wedge (R + x)\right)\vec{u}_\rho.$$

$$\vec{a}_e = -\omega_0^2 \left(R + \frac{1}{2}a_0 t^2\right)\vec{u}_\theta.$$

The entrainment acceleration corresponds to the acceleration of point P relative to R_T if it were fixed in the bus at that moment. It would then have a uniform circular motion with radius $(R+x)$ and angular velocity ω_0, resulting in an acceleration vector that is normal to the trajectory and centripetal.

$$\vec{a}_T = \vec{a}_r + \vec{a}_e + \vec{a}_c \Rightarrow \text{avec} \ \left(a_0 - \left(R + \frac{1}{2}a_0 t^2\right)\omega_0^2\right)\vec{u}_\rho + 2a_0 t\omega_0 \vec{u}_\theta.$$

4)

$$AB = 3\,m = \frac{1}{2}a_0 t^2$$

$$\Rightarrow \qquad t = \sqrt{\frac{2AB}{a_0}} = 1\,s \qquad v_b = a_0 t = 6\,m.s^{-1}.$$

The bus has turned by an angle $\theta = \omega_0 t = \frac{1}{6}\,rad = 9.55°$ and has traveled a distance $l = R\theta = 20\,m$..

Exercise 1.4

1) The velocity of M in R_1 is given by:

$$\overrightarrow{O_1M} = l(\cos\varphi\,\vec{\imath} + \sin\varphi\,\vec{\jmath})$$

$$\Rightarrow \vec{v}(M/R_1) = \left.\frac{d\overrightarrow{O_1M}}{dt}\right|_{R_1} = \boxed{l\dot{\varphi}(-\sin\varphi\,\vec{\imath} + \cos\varphi\,\vec{\jmath})}$$

The acceleration of M in R_1 is expressed as follows:

$$\Rightarrow \vec{\gamma}(M/R_1) = \left.\frac{d\vec{v}(M/R_1)}{dt}\right|_{R_1}$$

$$= \boxed{-l(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi)\vec{\imath} + l(\ddot{\varphi}\cos\varphi + \dot{\varphi}^2\sin\varphi)\vec{\jmath}}$$

2) The reference frame R_1 is in translation relative to the reference frame R, hence $\vec{\omega}(R_1/R) = \vec{0}$ and the entrainment velocity is given by:

$$\vec{v}_e = \left.\frac{d\overrightarrow{OO_1}}{dt}\right|_R = \dot{y}\vec{j}$$

The entrainment acceleration is:

$$\vec{\gamma}_e = \left.\frac{d^2\overrightarrow{OO_1}}{dt^2}\right|_R = \ddot{y}\vec{j}$$

The Coriolis acceleration is zero since R_1 is in translation relative to R.

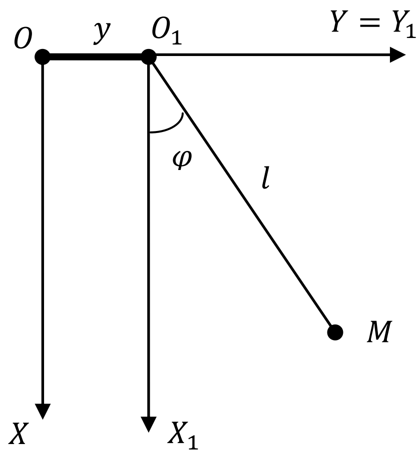
3) La vitesse de M dans R est donnée par:

$$\vec{v}(M/R) = \vec{v}(M/R_1) + \vec{v}_e = \boxed{-l\dot{\varphi}\sin\varphi\,\vec{\imath} + (l\dot{\varphi}\cos\varphi + y)\vec{\jmath}}$$

The acceleration of M in R is equal to

$$\vec{\gamma}(M/R) = \left.\frac{d^2\overrightarrow{OO_1}}{dt^2}\right|_R + \vec{\gamma}(M/R_1)$$

$$\Longrightarrow \boxed{\vec{\gamma}(M/R) = -l(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi)\vec{\imath} + [\ddot{y} + l(\ddot{\varphi}\cos\varphi - \dot{\varphi}^2\sin\varphi)]\vec{\jmath}}$$



Chapitre III

Dynamics of a Particle

III- Dynamics of a Particle in a Galilean Reference Frame

Dynamics is the study of motion in relation to the causes that produce it. These causes are the interactions between particles and are represented by forces.

3.1 Momentum

Let A be a particle with mass m moving with respect to a reference frame R. The momentum of A with respect to R is a vector denoted by $\vec{P}_{A/R}$ and defined as:

$$\vec{P}_{A/R} = m.\vec{v}_{A/R}$$

Unit: kg.m/ s; dimension: [momentum] = MLT^{-1}

3.2 Fundamental Laws of Dynamics

3.2.1 Galilean Reference Frame - Principle of Inertia

- **Statement:**
  A Galilean reference frame is a reference frame in which any isolated point is either at rest or moving with a constant velocity.
- **Property:**
  If R is a Galilean reference frame, then any reference frame that is uniformly translating relative to R is also Galilean.
- **Examples of Galilean reference frames:**
  - *Copernican Reference Frame: This reference frame has its origin at the barycenter of the solar system and its axes directed towards three distant stars (sidereal reference frame). This reference frame is Galilean.*
  - *Geocentric Reference Frame: This reference frame has its origin at the barycenter of the Earth and its axes parallel to those of the Copernican*

reference frame. The geocentric reference frame undergoes an elliptical translation motion relative to the Copernican reference frame.

- *Earth Reference Frame It is a reference frame attached to the Earth. It is not strictly Galilean due to the Earth's rotation on its axis and around the Sun. However, it can be approximated as a Galilean reference frame with less accuracy.*

3.2.2 Principle of Inertia:

In a Galilean reference frame, any mechanically isolated particle is either at rest or moving with uniform rectilinear motion.

3.2.3 Fundamental Relation of Dynamics (FRD) (or Kinetic Resultant Theorem)

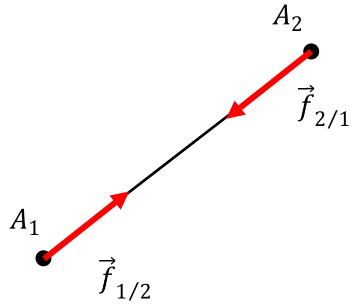
- **Statement:** In a Galilean reference frame *R*, the motion of a particle A with mass *m*, subjected to a resultant force $\sum f$, is governed by the following vectorial relation:

$$\left.\frac{d\vec{P}_{A/R}}{dt}\right|_R = \sum \vec{f} \; avec \; \vec{P}_{A/R}: \; represents\ the\ momentum\ vector\ of\ A\ in\ R.$$

Unit of force: N (Newton) ; $1\,N = 1\,kg.m.s^{-2}$ *in S.I*
Dimension: $[force] = MLT^{-2}$.

a. Principe de l'action et de la réaction

Given that A_1 and A_2 two material points, $\vec{f}_{1/2}$ and $\vec{f}_{2/1}$ the reciprocal intercation forces, applied by A_1 on A_2 and that applied by A_2 on A_1 respectively.



$\vec{f}_{1/2}$ and $\vec{f}_{2/1}$ are as follow :

- $\vec{f}_{1/2}$ and $\vec{f}_{2/1}$ are opposite and belong in the same sgment $[A_1\ A_2]$.
- $\vec{f}_{1/2}$=- $\vec{f}_{2/1}$ et $\vec{f}_{1/2} \wedge \overrightarrow{A_1A_2} = \vec{0}$

In the case of the system { A_1, A_2} is isolate est isolé (in R reference frame):

$$\vec{f}_{1/2} = \left.\frac{d\vec{P}_{A_1/R}}{dt}\right|_R \quad et \;\; \vec{f}_{2/1} = \left.\frac{d\vec{P}_{A_2/R}}{dt}\right|_R \quad \Rightarrow \quad \frac{d}{dt}\left(\vec{P}_{A_1/R} + \vec{P}_{A_2/R}\right) = \vec{0}$$

$$\Rightarrow \vec{P}_{\{A_1,A_2\}} = Constant.$$

The momentum, with respect to the reference frame of isolated system, is conserved.

3.3 Angular momentum

3.3.1 Definition :

Given A material point, with m masse, moving with respect to an arbitrary reference frame. Given that Q a geometrical point, the angular momentum, in Q of A regarding R is a vector denoted $\vec{L}_{Q/R}$ defined by:

$$\vec{L}_{Q/R} = \overrightarrow{QA} \wedge \vec{P}_{A/R}$$

Unit: $kg.m^2.s^{-1}$.
Dimension: $[angular\ momentum] = ML^2T^{-1}$.

3.3.2 Theorem of Angular Momentum (T.A.M)

Let A a material point, with m masse, moving reggarding a gallilian reference frame R . Let Q a geometrical point:

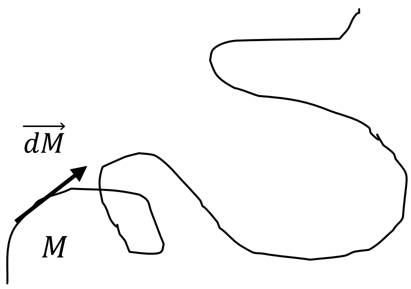
$$\left.\frac{d\vec{L}_{Q/R}}{dt}\right|_R = \overrightarrow{QA} \wedge \vec{P}_{A/R} + \overrightarrow{QA} \wedge \sum \vec{f}$$

$\overrightarrow{QA} \wedge \sum \vec{f} = \vec{M}_Q(\sum \vec{f})$: momentum vector, the forces applied on A, in Q point.

If $Q = 0$ fixed with respect to R: $\left.\frac{d\vec{L}_{Q/R}}{dt}\right|_R = \overrightarrow{OA} \wedge \sum \vec{f} = M_0(\sum \vec{f})$

In a galilian reference frame, the derivative regading time of angular momentum of a material point, realted to a fixe point O, is equal to the moment of the forces applied on A.

3.4 Kinetic Energy - Potential Energy - Mechanical Energy



3.4.1 Work

Let M a point belongs to R contrained by force $\vec{f}$. We define the elementary work of the force $\vec{f}$ while the displacement $\overrightarrow{dM}$ by:

$$\delta W = \vec{f}.\overrightarrow{dM}$$

We have therefore:

$$W = \int_{M_1}^{M_2} \vec{f}.\overrightarrow{dM} = \int_{t_1}^{t_2} \vec{f}.\vec{v}dt$$

The kinetic energy of a material point with m masse and a velocity $\vec{v}$ is defined by:

$$E_c = \frac{1}{2}mv^2$$

3.4.2 Theorem of Kinetic Energy (T.K.E)

The variation of the kinetic energy of a point between two instants in a reference frame is equal to the forces work applied on the point between the two instants:

$$E_{c_2} - E_{c_1} = W_{1\to2}$$

This theorem is very useful for resolving the problems of one freedom degree. We can also use it under another aspect, i.e. theorem of the kinetic power

$$\frac{dE_c}{dt} = \vec{f}.\vec{v}$$

Warning: *if mass of the system vary, the theorem of kinetic energy (T.K.E) couldn't be applied. indeed, its demonstrating starting with Fundamental Relation of Dynamics (FRD), suppose that the m masse of the system is constant.*

3.4.3 Conservative Force - Potential Energy

We can say that the f force is conservative if a scalar function U existe and defined as : $\vec{f} = -\overrightarrow{grad}U$. We can then say that the $\vec{f}$ force is derived of a potential, or potential energy U.

If we find the work of this $\vec{f}$ force while de displacement $M_1 \to M_2$, we have:

$$W = \int_{M_1}^{M_2} \vec{f}.\overrightarrow{dM} = -\int_{M_1}^{M_2} \overrightarrow{grad}U.\overrightarrow{dM} = -\int_{M_1}^{M_2} dU \quad d'après\ la\ propriété\ \ du\ graduit. Alor:$$

$$W = U(M_1) - U(M_2) = -\Delta E_p$$

The work of a conservative force while a displacement $M_1 \rightarrow M_2$ don't depends of the followed path but only on the start point and the end point : is equal to the decreasing of the potential energy.

3.4.4 Mechanical Energy

When we are in presence of the conservatives forces, the Theorem of Kinetic Energy is written as :

$$\Delta E_c = W = -\Delta U \quad \Leftrightarrow \Delta(E_c + U) = 0$$

We call mechanical energy the quantity :

$$E_m = E_c + U$$

If the system is only constrained by concervatives forces, its mechanical energy will conserved along the time.

Exercices

Dynamics of a Particle

III- Dynamics of a Particle

Exercise 1.1

We have two identical linear springs with a rest length of l. Each spring is subjected to a weight vector $\vec{P}_0$, causing an elongation of l_0 determined by their common stiffness k. We suspend a weight P_0 on one of the springs and horizontally pull the weight using the other spring with a variable force vector $\vec{F}$. The first spring then forms an angle α with the vertical.

- For each value of α corresponding to a force $\vec{F}$, the spring (1) undergoes an elongation l_1, and the spring (2) undergoes an elongation l_2.
- Calculate the elongations l_1 and l_2 as functions of α and l_0.

Exercise 1.2

A particle M with charge q and mass m is subjected to the action of a constant magnetic field $\vec{B}$.

Let $R(O, X, Y, Z)$ be a Galilean reference frame with the orthonormal direct basis $(\vec{\imath}, \vec{\jmath}, \vec{k})$, such that $\vec{B} = B\vec{k}$. The initial velocity of the particle is $\vec{v}_0 = \vec{v}_{0\perp} + \vec{v}_{0\parallel}$, where $\vec{v}_{0\perp} = \vec{v}_{0x}\vec{\imath} + \vec{v}_{0y}\vec{\jmath}$ is the projection of the velocity onto the *(OXY)* plane, and $\vec{v}_{0\parallel} = \vec{v}_{0z}\vec{k}$ is the component of the velocity parallel to the magnetic field. Let $\omega_c = qB/m$. We neglect the effect of gravity compared to the effect of the magnetic field.

1. Provide the expression for the force $\vec{F}_B$ exerted on the particle by the magnetic field.
2. Apply the Newton's second law and show that $\vec{v}_\parallel$ is a constant of motion.
3. Express $\left.\frac{d\vec{v}(M/R)}{dt}\right|_R$ and deduce that the magnitude $v = \|\vec{v}(M/R)\|$ is constant. Consequently, $\vec{v}_\perp$ is also a constant of motion.
4. Project the Newton's second law onto the Cartesian basis and derive the differential equations for v_x, v_y and v_z.
5. Solve the previous equations and show that the time-dependent equations of motion are:

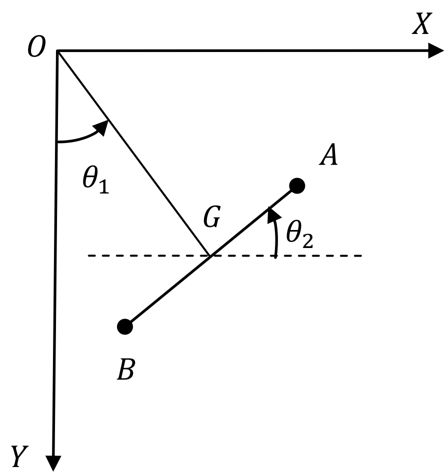
$$\begin{cases} x = \dfrac{\vec{v}_{0\perp}}{\omega_c}\sin\omega_c t \\ y = \dfrac{\vec{v}_{0\perp}}{\omega_c}(1-\cos\omega_c t) \\ z = \vec{v}_{0\parallel} t \end{cases}$$

6. What is the nature of the trajectory?

Exercise 1.3

Two identical balls, considered as point masses of mass *m*, are fixed at the two ends of a rod *AB* with negligible mass and length $2d$. This rod, constrained to stay in the *(OX, OY)* plane, is hinged at *G* to a slender rod *OG* with negligible mass and length *a*. The motion is characterized by the angles θ_1 and θ_2 (see figure).

1. Calculate directly the angular momentum $\vec{L}_0$ of the system with respect to point *O* in terms of *m*, *a*, *l*, θ_1, and θ_2.



Exercise 1.4

A particle of mass *m* moves under the action of an attractive force given by $\vec{F} = \frac{k}{r^2}\vec{u}$, where *r* is the distance from the particle to the origin. The trajectory is a circle of radius *r*. Show that:

1. The total energy is $E = -\frac{k}{2}$.
2. The velocity is $v = \sqrt{\frac{k}{m}}$.
3. The angular momentum is $L = \sqrt{mkr}$.

Solution

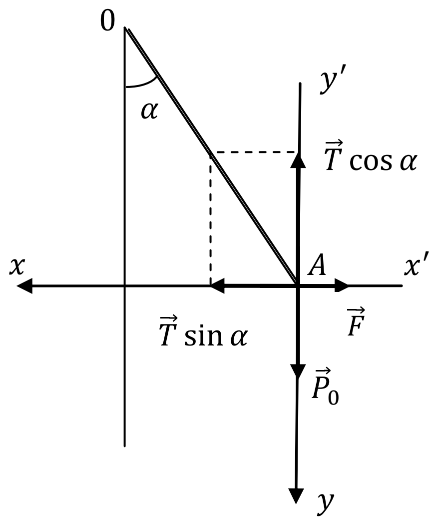
Exercise 1.1

We have represented in the figure below the forces acting on point A. By projecting onto the two mutually perpendicular axes, we obtain the following equilibrium:

$$\left.\begin{array}{r} P_0 = T\cos\alpha \\ P_0 = kl_0 \\ T = kl_1 \end{array}\right| \Rightarrow \boxed{l_1 = \frac{l_0}{\cos\alpha}}$$

$F = T\sin\alpha$

$$\left.\begin{array}{l} F = kl_2 = T\sin\alpha \\ T = \dfrac{P_0}{\cos\alpha} = \dfrac{kl_0}{\cos\alpha} \end{array}\right| \Rightarrow \frac{kl_0}{\cos\alpha}\sin\alpha = kl_2 \Rightarrow \boxed{l_2 = l_0\tan\alpha}$$



Exercise 1.2

In the following expressions, let $\vec{V}(M/R)$ be denoted as $\vec{v}$.

1. The expression for the force $\vec{F}_B$ is given by:

$$\vec{F}_B = q\vec{v}\wedge\vec{B}.$$

2. Since we neglect the weight of the charged particle, the only force acting on M is $\vec{F}_B$. R is Galilean, so the application of Newton's second law in $\vec{F}_B$ yields:

$$\vec{F}_B = m\left.\frac{d\vec{v}}{dt}\right|_R \Rightarrow \left.\frac{d\vec{v}}{dt}\right|_R = -\frac{qB}{m}\vec{k}\wedge\vec{v}.$$

Let $\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$ with $v_z = \vec{v}_{\parallel}$, then

$$\vec{k}\wedge\vec{v} = \vec{k}\wedge\left(v_x\vec{i} + v_y\vec{j} + v_{\parallel}\vec{k}\right)$$

$$= v_x\vec{j} - v_y\vec{i}$$

$$\vec{k} \wedge \vec{v} = \vec{k} \wedge \left(v_x\vec{i} + v_y\vec{j} + v_{\parallel}\vec{k}\right)$$

$$\Longrightarrow \left.\frac{d\vec{v}}{dt}\right|_R = \dot{v}_x\vec{i} + \dot{v}_y\vec{j} + \dot{v}_z\vec{k}$$

$$= -\frac{qB}{m}\left(v_x\vec{j} - v_y\vec{i}\right) \implies \dot{v}_z = 0 \implies v_z = v_{\parallel} = Constant$$

3. We have established that:

$$\left.\frac{d\vec{v}}{dt}\right|_R = -\frac{qB}{m}\vec{k} \wedge \vec{v}$$

where the derivative of any vector $\vec{A}$ in R with $\vec{u}_A = \vec{A}/\|\vec{A}\|$ is given by:

$$\left.\frac{d\vec{A}}{dt}\right|_R = -\frac{d\|\vec{A}\|}{dt}\vec{u}_A + \Omega(\vec{A}/R) \wedge \vec{A}.$$

Applying this result to:

$$\left.\frac{d\vec{v}}{dt}\right|_R = -\frac{qB}{m}\vec{k} \wedge \vec{v}$$

we deduce that $\frac{d\|\vec{v}\|}{dt} = 0 \implies \|\vec{v}\| = v$ is constant and $\vec{v}$ est rotates in R with the rotational vector $-\frac{qB}{m}\vec{k}$.

Since $v^2 = v_{\perp}^2 + v_{\parallel}^2$, on one hand, and v and $v_{\parallel}$ are constants, on other hand, $\vec{v}_{\perp}$ is constant.

4. Let's expand the equations in the Cartesian basis, considering that $\vec{B} = B\vec{k}$:

$$m\frac{dv_x}{dt}\vec{i} + m\frac{dv_y}{dt}\vec{j} + m\frac{dv_z}{dt}\vec{k} = q\left(v_x\vec{i} + v_y\vec{j} + v_z\vec{k}\right) \wedge B\vec{k}$$

$$= qBv_x\vec{i} - qBv_x\vec{j}$$

That gives

$$\begin{cases} \dfrac{dv_z}{dt} - \dfrac{qB}{m} v_y = 0 \\ \dfrac{dv_y}{dt} - \dfrac{q}{m} v_x = 0 \\ \dfrac{dv_z}{dt} = 0 \end{cases}$$

We observe that the first two equations are coupled, meaning that v_x and v_y appear in both equations. The component of velocity along O_z is constant and equal to the component of velocity along O_z at $t = 0$, $v_z = v_{0\parallel}$.

5. Let's revisit the previous equations by replacing v_$v_x = \dot{x}$, $v_y = \dot{y}$ and $v_z = \dot{z}$ ˙, which gives:

   $z = v_{0\parallel} t + K$ avec $K = z(0) = 0$

Similarly,

$$\ddot{y} = -\frac{qB}{m}\dot{x} \quad \Longrightarrow \quad \dot{y} = -\omega_c x + K$$

$x(0) = 0$ and $\dot{y}_0 = 0$ then $K = 0$

$$\ddot{x} = \omega_c \dot{y} = -\omega_c^2 x \quad \Longrightarrow \quad \ddot{x} + \omega_c^2 x = 0$$

which is a second-order equation with constant coefficients and no forcing term. The solution is $x = Asin(\omega_c t - \varphi_0)$, with $x(0) = 0 = A \sin \varphi_0 \Longrightarrow \varphi_0 = 0$ $A \neq 0$, et $\dot{x}(0) = v_{0\perp} = A\omega_c \cos \varphi_0 \Longrightarrow A = v_{0\perp}/\omega_c$. Therefore, the solution becomes:

$$x(t) = \frac{v_{0\perp}}{\omega_c} \sin \omega_c t$$

The equation for y then becomes:

$$\dot{y} = -\omega_c x = -v_{0\perp} sin\omega_c t \quad \Longrightarrow y = +\frac{v_{0\perp}}{\omega_c} \cos \omega_c t + \mathrm{K}$$

Since $y(0) = 0 \quad \Longrightarrow K = -\frac{v_{0\perp}}{\omega_c}$ the final expression is :

$$\begin{cases} x = \dfrac{v_{0\perp}}{\omega_c} \sin \omega_c t \\ y = -\dfrac{v_{0\perp}}{\omega_c}(1 - \cos \omega_c t) \\ z = v_\parallel t \end{cases}$$

From the previous expressions, we have:

$$x^2 + \left(y + \frac{v_{0\perp}}{\omega_c}\right)^2 = \frac{{v_{0\perp}}^2}{{\omega_c}^2} \quad et\ z = v_{0\parallel}t$$

These equations represent a circle with its center at $\left(O, -\frac{v_{0\perp}}{\omega_c}\right)$and a radius of $-\frac{v_{0\perp}}{\omega_c}$ in the (Oxy) plane, and a uniform rectilinear translation along O_z. It is a helical motion, and the trajectory is a spiral around O_z.

Exercise 1.3

In the case of this system, the angular momentum of the system with respect to point O is equal to the sum of the angular momenta of all the individual components of the system. Specifically, the angular momentum of the system with respect to point O is equal to the angular momentum $\vec{L}_{O/G}$ of point G (which is the center of mass of the two masses) with respect to O, plus the angular momenta of points A ($\vec{L}_{A/G}$) and B ($\vec{L}_{B/G}$) with respect to G.

1. $\vec{L}_O = \vec{L}_{O/G} + \vec{L}_{A/G} + \vec{L}_{B/G}$

Let's start by calculating. ($\vec{L}_{O/G}$):

$$\left.\begin{aligned}\vec{L}_{G/O} &= \overrightarrow{OG} \wedge \vec{P}_{G/O} \\ \vec{P}_{G/O} &= 2m\vec{v}_{G/O}\end{aligned}\right| \Rightarrow \vec{L}_{G/O} = 2m\left(\overrightarrow{OG} \wedge \vec{v}_{G/O}\right)$$

$$\vec{L}_{G/O} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x_G = a\cos\theta_1 & y_G = a\sin\theta_1 & 0 \\ \dot{x}_G = -a\dot{\theta}_1\sin\theta_1 & \dot{y}_G = a\dot{\theta}_1\cos\theta_1 & 0 \end{vmatrix} = (x_G\dot{y}_G - \dot{x}_G y_G)$$

$$\boxed{\vec{L}_{G/O} = 2ma^2\dot{\theta}_1^2} \qquad \rightarrow \qquad (1)$$

Next, let's calculate:

($\vec{L}_{A/G} = \vec{L}_{B/G}$):

$$\left.\begin{aligned}\vec{L}_{A/G} &= \overrightarrow{GA} \wedge \vec{P}_{A/G} \\ \vec{P}_{A/G} &= m\vec{v}_{A/G}\end{aligned}\right| \Rightarrow \vec{L}_{A/G} = m\left(\overrightarrow{GA} \wedge \vec{v}_{A/G}\right)$$

$$\vec{L}_{O/G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_A = d\cos\theta_2 & y_A = d\sin\theta_2 & 0 \\ \dot{x}_A = -d\dot{\theta}_2\sin\theta_2 & \dot{y}_A = d\dot{\theta}_2\cos\theta_2 & 0 \end{vmatrix} = (x_A\dot{y}_A - \dot{x}_A y_A)$$

$$\boxed{\vec{L}_{A/G} = md^2\dot{\theta}_2^2 = \vec{L}_{B/G}} \quad \rightarrow \quad (2)$$

Finally, we just need to add the expressions (1) and (2) to find the answer to the question.

$$\vec{L}_O = 2ma^2\dot{\theta}_1^2 + 2md^2\dot{\theta}_2^2$$

$$\Longrightarrow \boxed{\vec{L}_O = 2m\left(a^2\dot{\theta}_1^2 + d^2\dot{\theta}_2^2\right)}$$

Exercise 1.4

1-Since the force is central and only depends on the radial distance r, its potential energy exhibits spherical symmetry and varies only with r. The relationship between the force and the potential energy is given by $\vec{F} = -\vec{\nabla}E_P$. Since r is the only variable, this relationship holds true solely in the radial component, $dE_P/dr = k/r^2$. From this, we can deduce the value of the potential energy:

$$E_P = \int \frac{k}{r^2}dr \quad \Longrightarrow \quad E_P = -\frac{k}{r} + C^{te}$$

To determine the integration constant, we consider that E_P approaches 0 ($E_P = 0$) as r approaches infinity ($r \rightarrow \infty$). Therefore, the integration constant must be $C^t = 0$ Hence:

$$E_P = -\frac{k}{r} \quad \rightarrow \quad (1)$$

The total energy E is the mechanical energy, which is the sum of the potential energy E_P and the kinetic energy E_C.

Since the motion is circular and the trajectory is a circle, we have $v = \dot{\theta}$, where $\dot{\theta}$ represents the angular velocity.

Therefore:

$$E_C = \frac{1}{2}mv^2 \quad v = \dot{\theta}r \quad \Rightarrow E_C = \frac{1}{2}\underbrace{m\dot{\theta}^2 r}_{F=F_C} r$$

$$E_C = \frac{1}{2}\frac{k}{r^2}r \quad \Rightarrow \boxed{E_C = \frac{1}{2}\frac{k}{r}} \quad \rightarrow (2)$$

2- By adding equations (1) and (2) term by term, we obtain the total energy:

$$E = \frac{1}{2}\frac{k}{r} - \frac{k}{r} \quad \Rightarrow \boxed{E = -\frac{1}{2}\frac{k}{r}}$$

3- From equation (2), we can deduce the expression for the velocity:

$$E_C = \frac{1}{2}\frac{k}{r} = \frac{1}{2}mv^2 \quad \Rightarrow v = \sqrt{\frac{k}{mr}}$$

4- c/ Calculating the angular momentum in cylindrical coordinates with respect to the center of the circle:

$$\vec{L}_O = \vec{r} \wedge m\vec{v} \quad \vec{v} = \vec{v}_r + \vec{v}_\theta = r\dot{\theta}\vec{u}_\theta \quad \Rightarrow m\begin{vmatrix} \vec{u}_r & -\vec{u}_\theta & \vec{u}_z \\ r & 0 & 0 \\ 0 & r\dot{\theta} & 0 \end{vmatrix} \Rightarrow \vec{L}_O = mr^2\dot{\theta}\vec{u}_z$$

The magnitude of the angular momentum is therefore equal to:

$$L_O = mr^2\dot{\theta} \quad \dot{\theta} = \frac{v}{r} = \frac{1}{r}\sqrt{\frac{k}{mr}} \quad \Rightarrow L_O = mr^2\frac{1}{r}\sqrt{\frac{k}{mr}} \quad \Rightarrow \boxed{L_O = \sqrt{mkr}}$$

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