Part 1

29/10/2023

Chapter 3

Sequence of real numbers

1 Generalities

Definition 1.1 *A sequence of real numbers is a function*

$$U : \begin{cases} \mathbb{N} \to \mathbb{R} \\ n \mapsto u(n) = u_n \end{cases}.$$

Informally,the sequence U can be written as an infinite list of real numbers as

$U = (u_1, u_2, u_3, ...)$, where $u_n = U(n)$.

Other notations for sequences are (u_n) or $\{u_n\}_1^{\infty}$; we will use (u_n).

Some sequences can be written **explicitly** with a formula such as

$$u_n = \frac{n}{n+1}, u_n = \frac{1}{2^n} \text{ or } u_n = (-1)^n \cos(n^2+1)$$

or we could be given the first few terms of the sequence, such as

the other may be given **recursively, for example** $\begin{cases} u_0 = 1 \\ u_{n+1} = \sqrt{2+u_n} \end{cases}, \forall n \in \mathbb{N}$

with $u_0 = 1, u_1 = \sqrt{3}, u_2 = \sqrt{2+\sqrt{3}}, ..$

1.1 Classical sequences

There are two classical sequences that we will encounter quite often. Arithmetic sequences and geometric sequences..

Arithmetic sequences

An arithmetic sequence is a sequence $(u_n)_{n\in\mathbb{N}}$ for which there exists $a \in \mathbb{R}$ called the common difference of this sequence such that, for all $n \in \mathbb{R}$,

$$u_{n+1} = a + u_n. \qquad \textbf{Recurrent form}$$

The general term of an arithmetic sequence with common difference a and first term u_0 is

$$u_n = u_0 + na. \qquad \textbf{Explicit form}$$

Geometric sequences

A geometric sequence is a sequence $(u_n)_{n\in\mathbb{N}}$ for which there exists $r \in \mathbb{R}$ called the common ratio of this sequence such that, for all $n \in \mathbb{N}$

$$u_{n+1} = ru_n. \qquad \textbf{Recurrent form}$$

The general term of a geometric sequence with common ratio r and first term u_0 is

$$u_n = u_0 r^n. \qquad \textbf{Explicit form}$$

Proposition 1.1 *: Let $u(n)$ and $v(n)$ be two real sequences, and let :λ be a real number. We define the sequences :*

▷ 1) $x = u + v$ with general term $x_n = u_n + v_n$
▷ 2) $w = uv$ with general term $w_n = u_n v_n$
▷ 3) $y = \lambda u$ with general term $y_n = \lambda u_n$

$\triangleright$ 4) $\forall n \in \mathbb{N}$, $v_n \neq 0$, we define $z = \left(\frac{u}{v}\right)$ with general term $z_n = \frac{u_n}{v_n}$.

2 Monotone Sequences, Boundedness

Definition 2.1 *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence .*

*i) We say that $(u_n)_{n\in\mathbb{N}}$ is **increasing** if $u_n \leq u_{n+1}$, for all $n \in \mathbb{N}$*

*ii) We say that $(u_n)_{n\in\mathbb{N}}$ is **decreasing** if $u_{n+1} \leq u_n$, for all $n \in \mathbb{N}$.*

*iii) We say that $(u_n)_{n\in\mathbb{N}}$ is **monotone** if $(u_n)_{n\in\mathbb{N}}$ is either **increasing** or **decreasing**.*

iv) $(u_n)_{n\in\mathbb{N}}$ is a constant sequence if and only if $u_n = u_{n+1}$ for all $n \in \mathbb{N}$.

Remark To study the variation of a sequence $(u_n)_{n\in\mathbb{N}}$, we compare the terms u_{n+1} and u_n for each integer n, either by studying the sign of the difference $u_{n+1} - u_n$, or, when the terms u_n are strictly positive, by comparing the ratio $\dfrac{u_{n+1}}{u_n}$ to the number 1.

Example 2.1 *let $u_n = 3n + 2, \forall n \in \mathbb{N}$.*

We have $u_{n+1} - u_n = 3(n+1) + 2 - (3n+2) = 3n + 3 + 2 - 3n - 2 = 3 > 0$ therefore $(u_n)_{n\in\mathbb{N}}$ is **increasing**.

Example 2.2 *for all $n \geq 1$, $u_n = \dfrac{n}{2^n}$.*

we have $\dfrac{u_{n+1}}{u_n} = \dfrac{n+1}{2^{n+1}}\dfrac{2^n}{n} = \dfrac{1}{2}\dfrac{n+1}{n} = \dfrac{1}{2}\left(1 + \dfrac{1}{n}\right) \leq \dfrac{1}{2}(1+1) = 1$

Then $\dfrac{u_{n+1}}{u_n} \leq 1 \Rightarrow u_{n+1} \leq u_n, \forall n \geq 1$. *so* $(u_n)_{n\in\mathbb{N}}$ *is* **decreasing**.

2.1 Upper and lower bound sequences

Definition 2.2 *let $(u_n)_{n\in\mathbb{N}}$ be a sequence*

- $(u_n)_{n\in\mathbb{N}}$ is **upper bounded** if $\exists M \in \mathbb{R}, \ \forall n \in \mathbb{N}, \ u_n \leq M$.
- $(u_n)_{n\in\mathbb{N}}$ **is lower bounded** if $\exists m \in \mathbb{R}, \ \forall n \in \mathbb{N}, \ u_n \geq m$
- $(u_n)_{n\in\mathbb{N}}$ **is bounded** if (she is upper or lower bounded) $\exists M, m \in \mathbb{R}, \quad m \leq u_n \leq M$.

let $(u_n)_{n\in\mathbb{N}} = (\sin(n))_{n\in\mathbb{N}}$ is bounded when $\forall n \in \mathbb{N} : |\sin(n)| \leq 1$ so $\forall n \in \mathbb{N} : -1 \leq \sin(n) \leq 1$

2.2 Monotone Convergence Theorem

Theorem 2.1 *If (u_n) is bounded and monotone then (u_n) is convergent.In particular;*

i) if (u_n) is bounded above and increasing then $\lim\limits_{n\to+\infty} u_n = \sup\{u_n : n \in \mathbb{N}\}$,

ii) if (u_n) is bounded below and decreasing then $\lim\limits_{n\to+\infty} u_n = \inf\{u_n : n \in \mathbb{N}\}$.

the results mentioned previously, which apply to bounded monotone sequences, can also be applied to unbounded monotone sequences.

Example : The sequence $(\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, ...)$ is decreasing and unbounded. It converges to 0

Theorem 2.2 *Any unbounded increasing sequence tends to $+\infty$. Any unbounded decreasing sequence tends to $-\infty$*

Example 2.3 *let $u_n = \frac{1}{n}$,*

we have $\forall n \in \mathbb{N}^* : 0 < \frac{1}{n} \leq 1$.

Moreover, we have $\frac{u_{n+1}}{u_n} = \frac{n}{n+1} < 1$, so (u_n) is decreasing and bounded below by 0, so (u_n) convergences.

3 Limits of sequences

Definition 3.1 *A sequence of real numbers $(u_n)_{n\in\mathbb{N}}$ is said to converge to a real number L,if*

$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N} : (n \geq N \Longrightarrow |u_n - L| < \varepsilon)$.*and we writre* $\lim\limits_{n\to+\infty} u_n = L \Longleftrightarrow \lim\limits_{n\to+\infty} (u_n - L) = 0 \Longleftrightarrow \lim\limits_{n\to+\infty} |u_n - L| = 0$.

The number L is called the limit of the sequence. If (u_n) converges to L we will write

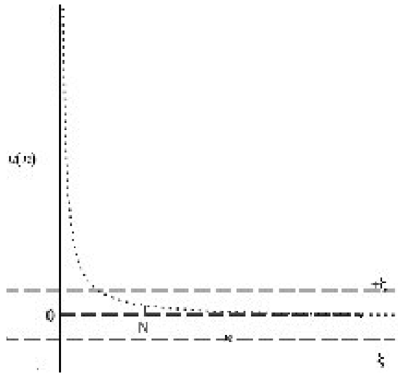
$\triangleright \lim\limits_{n \to +\infty} u_n = L, \exists N \in \mathbb{N}, \forall n \in \mathbb{N} : (n \geq N \Longrightarrow |u_n - l| < \varepsilon$

• *If a sequence does not converge, then we say that it **diverges**.*

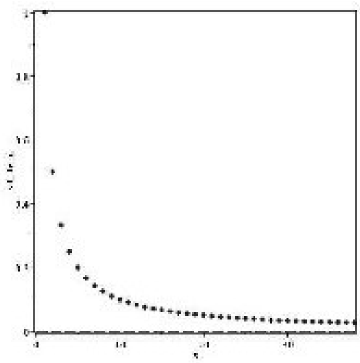
Example : Using the definition of the limit of a sequence, prove that $\lim\limits_{n \to +\infty} \frac{1}{n} = 0.$

Take any $\varepsilon > 0.$ *Then* $\exists N \in \mathbb{N}, \forall n \in \mathbb{N} : (n \geq N \Longrightarrow \left|\frac{1}{n} - 0\right| < \epsilon \Rightarrow \frac{1}{n} < \varepsilon$, if $n \geq N$ then $\frac{1}{n} \leq \frac{1}{N} < \varepsilon.$

$\left(\text{by Archimedean property : } \exists N \in \mathbb{N}^* \text{such that } N > \frac{1}{\varepsilon}\right)$ This proves, by definition, that $\lim\limits_{n \to +\infty} \frac{1}{n} = 0$; so a sequence $\left(\frac{1}{n}\right)$ converges to 0.



cvg



1.*jpg*

02/11/2023

Example 3.1 *Using the definition of the limit of a sequence, prove that*

$$\lim_{n \to +\infty} \frac{n}{2n+1} = \frac{1}{2}$$

proofet $\varepsilon > 0$. we must show that there is $N \in \mathbb{N}$,such that $\forall n \in \mathbb{N} : (n \geq N$ implies $\left|\frac{n}{2n+1} - \frac{1}{2}\right| < \varepsilon$ **1.**

Some work with the expression in absolute values shows us how to do this

$$\left|\frac{n}{2n+1}-\frac{1}{2}\right|=\left|\frac{2n-2n+1}{4n+2}\right|=\left|\frac{1}{4n+2}\right|=\frac{1}{4n+2}<\frac{1}{4n}.$$

Thus $\left|\frac{n}{2n+1}-\frac{1}{2}\right|<\varepsilon$ whenever $\frac{1}{4n}.<\varepsilon$ which gives (by archimedean property) $n>\frac{1}{4\varepsilon}$.Thus, it suffices to choose $N=E\left(\frac{1}{4\varepsilon}\right)+1$

Exercise Using the definition of the limit of a sequence, show that the following sequences converges to l

- $(u_n)_{n\in\mathbb{N}}$; $u_n=\frac{(2n^2+1)^2}{n^4}$, $l=4$
- $(u_n)_{n\in\mathbb{N}}$; $u_n=\sqrt[n]{a}, a>1$; $l=1$

Definition 3.2 *We say that the sequence* $(u_n)_{n\in\mathbb{N}}$ *tends to* $+\infty$ *as* n *tends to infinity and we note* $\lim\limits_{n\to+\infty} u_n=+\infty$ *iff*

$\triangleright$ $\forall A>0, \exists N\in\mathbb{N}, \forall n\in\mathbb{N}:(n\geq N\Longrightarrow u_n\geq A$

In this case we will say that (u_n) **diverges** to $+\infty$. We can make a similar definition for $\lim\limits_{n\to+\infty} u_n=-\infty$.

$\triangleright$ $\forall A>0, \exists N\in\mathbb{N}, \forall n\in\mathbb{N}:(n\geq N\Longrightarrow u_n\leq -A$

Example 3.2 *Let be the following sequences*

$U_n=\frac{3}{2}n^2+1,\quad V_n=-3n+5$

We show that $\lim\limits_{n\to+\infty} U_n=+\infty$ and $\lim\limits_{n\to+\infty} V_n=-\infty$.

1• $\lim\limits_{n\to+\infty} U_n=+\infty$, using the definition of limit

$\triangleright$ $\forall A>0, \exists N\in\mathbb{N}, \forall n\in\mathbb{N}:(n\geq N\Longrightarrow u_n\geq A$

Let $A>0$, we have $u_n\geq A\Rightarrow \frac{3}{2}n^2+1\geq A$ implies $n^2\geq\frac{2(A-1)}{3}$

we obtain $n\geq\sqrt{\frac{2(A-1)}{3}}$, so, $\exists N$, we can take $N=E\left[\sqrt{\frac{2(A-1)}{3}}\right]+1$

• The same method used for the sequence $(V_n)_{n\in\mathbb{N}}$

3.1 Limit Theorems

Theorem 3.1 ***(Convergent sequences are bounded)*** *Let* (u_n)*,*$n \in N$ *be a convergent sequence. Then the sequence is bounded, and the limit is unique.*

proofEasier property to show that the limit is unique , so let's do that first. Suppose the sequence has two limits L and L'. **1.**

$\triangleright$*Take any* $\varepsilon > 0$*, Then* $\exists N \in \mathbb{N}, \forall k \in \mathbb{N} : (k \geq N \Longrightarrow |u_k - L| < \frac{\varepsilon}{2}$

Also, $\exists N' \in \mathbb{N}$ *another integer such that* $(\frac{\varepsilon}{2} \Longrightarrow |u_k - L'| < \frac{\varepsilon}{2}$

Then, by the triangle inequality :

$|L - L'| = |L - u_k + u_k - L| < |u_k - L| + |u_k - L'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ *if* $k \geq \max\{N, N'\}$

Therefore $|L - L'| < \varepsilon$ *,*$\forall \varepsilon > 0$.

But the only way that can happen is for $L = L'$*, so that the limit is indeed* **unique.**

$\triangleright$*Next, we need to prove boundedness. Since the sequence converges, we can take* $\forall \varepsilon > 0$

we *take* $\varepsilon = 1.(0 < \varepsilon \leq 1)$ *Then there is an integer* N *so that* $|u_{n_0} - L| < 1$ *if* $k > N$*.*

Fix that integer N*. Then we have that* $|u_n| \leq |u_n - L| + |L| < 1 + |L| = K$ *for all* $k > N$*.*

Now, define $M = max\{\{|u_k|, k = 1, ..., N\}, P\}$*. Then* $|u_n| < M$ *for all* n*, which makes the sequence* **bounded.**

Proposition 3.1 *Let* (u_n) *and* (v_n) *two convergent sequences*

› *if* $\lim_{n\to+\infty} u_n = l$*, for all* $\lambda \in \mathbb{R}$*, we have* $\lim_{n\to+\infty} (\lambda u_n) = \lambda l$*.*

› *if* $\lim_{n\to+\infty} u_n = l$ *and* $\lim_{n\to+\infty} v_n = l'$*, so* $\lim_{n\to+\infty} (u_n + v_n) = l + l'$ *and* $\lim_{n\to+\infty} u_n.v_n = l.l'$*.*

› *if* $\lim_{n\to+\infty} u_n = l$ *and* $l \neq 0$*, so* $\lim_{n\to+\infty} \frac{1}{u_n} = \frac{1}{l}$

› *if* $\lim_{n\to+\infty} u_n = l$*,so* $\lim_{n\to+\infty} |u_n| = |l|$.

Proposition 3.2 ***If the sequence*** (u_n) ***is bounded and the sequence*** (v_n) ***converges to 0 then the sequence*** $(u_n v_n)$ ***converges to*** 0.