Proposition 3.2 ***If the sequence*** (u_n) ***is bounded and the sequence*** (v_n) ***converges to 0 then the sequence*** $(u_n v_n)$ ***converges to*** 0.

Proposition 3.3 *Let* (u_n) *and* (v_n) *two sequence of real number. Suppose that* $\lim_{n\to+\infty} v_n = +\infty$.

› we have $\lim_{n\to+\infty} \frac{1}{v_n} = 0$
› If (u_n) is below bound so $\lim_{n\to+\infty} (u_n + v_n) = +\infty$
› If (u_n) is lower bound with a positive number so $\lim_{n\to+\infty} (u_n v_n) = +\infty$

3.2 Passage to the limit in inequalities

Let (u_n) and (v_n) be convergent sequences of real numbers. if $u_n \leq v_n, \forall n \in \mathbb{N}$, then $\lim_{n\to+\infty} u_n \leq \lim_{n\to+\infty} v_n$.
Important : The strict inequality is not preserved by passage to the limit, i.e. if $u_n < v_n, \forall n \in \mathbb{N}$, then $\lim_{n\to+\infty} u_n \leq \lim_{n\to+\infty} v_n$.
Exemple : Let $u_n = \dfrac{-1}{n}$ and $v_n = \dfrac{1}{n}, \forall n \in \mathbb{N}^*$, we have $u_n < v_n$ but $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} v_n = 0$

Corollary 3.1 *Let* (u_n) *be a convergent sequence of real numbers. If* $u_n \geq 0, \forall n \in \mathbb{N}$, *then* $\lim_{n\to+\infty} u_n \geq 0$.

Theorem 3.2 *(**Squeeze theorem or "Sandwich theorem"**) :Let* $(u_n), (v_n)$ *and* (w_n) *be convergent sequences of real numbers.*

› Suppose $u_n \leq v_n \leq w_n, \forall n \in \mathbb{N}$. if the sequences (u_n) and (w_n) are convergent and if

$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} w_n$, the sequence (v_n) is convergent and $\lim_{n\to+\infty} v_n = \lim_{n\to+\infty} w_n = \lim_{n\to+\infty} u_n$.
› If $u_n \leq v_n$ for *all* n and if $\lim_{n\to+\infty} u_n = +\infty$, then $\lim_{n\to+\infty} v_n = +\infty$.

Example Study the nature of $(u_n)_{n\in\mathbb{N}^*}$ defined by it's general term $u_n = \dfrac{E(na)}{n}$

Theorem 3.3 *Let $f : I \to \mathbb{R}$ be the function and (u_n) be convergent sequences whose terms all belong to I. If $\lim\limits_{n\to+\infty} u_n = l$ and if f is continuous at l, then $\lim\limits_{n\to+\infty} f(u_n) = f(l)$.*

proofet $\varepsilon > 0$. Since f is continuous at l, we can find a number $\alpha > 0$ such that $|f(x) - f(l)| < \varepsilon, \forall x \in I$ and $|x - l| < \alpha$. Since the sequence (u_n) has limit $l, \exists N$ such that $|u_n - l| < \alpha$ if $n \geq N$. For all these integers n, we then have $|f(u_n) - f(l)| < \varepsilon$. This show that we have $\lim\limits_{n\to+\infty} f(u_n) = f(l)$. **1.**
The following proposition, which completes the theorem, is proved by simply adapting the previous proof.

Proposition 3.4 *Let (u_n) be a sequence of real numbers , let f be a function , and let l a real numbers.*

If $\lim\limits_{n\to+\infty} (u_n) = +\infty$ *and if* $\lim\limits_{n\to\infty} f(x) = l$ (*or* $+\infty$), then $\lim\limits_{n\to+\infty} f(u_n) = l$ (*or* $+\infty$).
Indeterminate forms : : $(-\infty + \infty), (0 \times \infty), \left(\frac{\infty}{\infty}\right), \left(\frac{0}{0}\right), (1^\infty), (\infty^0)$ et (0^∞).

4 Adjacent sequences

Definition 4.1 *Let (u_n) and (v_n) be two sequences of real numbers. We say that (u_n) and (v_n) are adjacent if*

› (u_n) *is* ***increasing*** *and* (v_n)*' is* ***decreasing***

› $(\lim (u_n - v_n) \to 0)$.

Theorem 4.1 *two adjacent sequences (u_n) and (v_n) converge to the same limit L and $u_n \leq L \leq v_n$ for all $n \in \mathbb{N}$.*

proofy assumption, the sequence (u_n) is increasing and bounded from above by v_0(or v_1 or any v_k). Therefore it is convergent to some limit L such that $u_n \leq L$. Similarly,(v_n) is decreasing and bounded from below by u_0(or any u_k). Therefore it is convergent to some limit $L' \leq v_n$. Since both sequences are convergent, we can write $\lim\limits_{n\to\infty} (u_n - v_n) = \lim v_n - \lim u_n = L' - L = 0$. **1.**

Example $\forall n \in \mathbb{N}^*, u_n = 1 - \frac{1}{n}$ and $v_n = 1 + \frac{1}{n}$ are adjacent sequences

Example 4.1 *Let (u_n) and (v_n) Two sequences*

$$\forall n \in \mathbb{N}^*, u_n = 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!}, \quad \text{and} \quad v_n = u_n + \frac{1}{n!}.$$

We have

$$\begin{aligned} u_{n+1} - u_n &= \left(1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!} + \frac{1}{(n+1)!}\right) - \left(1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!}\right) \\ &= \frac{1}{(n+1)!} \geq 0, \end{aligned}$$

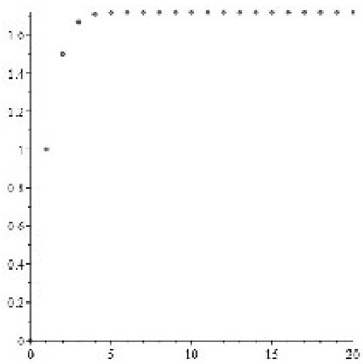
then (u_n) is increasing. Similarly

$$\begin{aligned} v_{n+1} - v_n &= \left(u_{n+1} + \frac{1}{n+1!}\right) - \left(u_n + \frac{1}{n!}\right) \\ &= (u_{n+1} - u_n) + \frac{1}{n+1!} - \frac{1}{n!} \\ &= \frac{1}{n+1!} + \frac{1}{n+1!} - \frac{1}{n!} \\ &= \frac{2}{n+1!} - \frac{1}{n!} \\ &= \frac{2}{(n+1)\,n!} - \frac{1}{n!} \\ &= \frac{2-n-1}{(n+1)\,n!} \\ &= \frac{1-n}{(n+1)\,n!} \leq 0. \end{aligned}$$

Then (v_n) is decreasing. We can write ,

$$\lim_{n\to\infty} (v_n - u_n) = \lim_{n\to\infty} \frac{1}{n!} = 0.$$

Then, (u_n) and (v_n) are adjacent.



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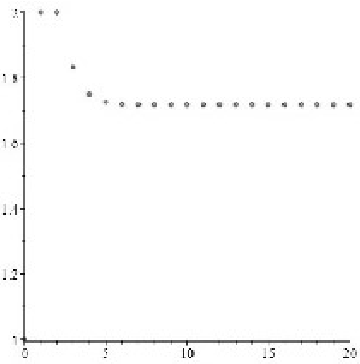

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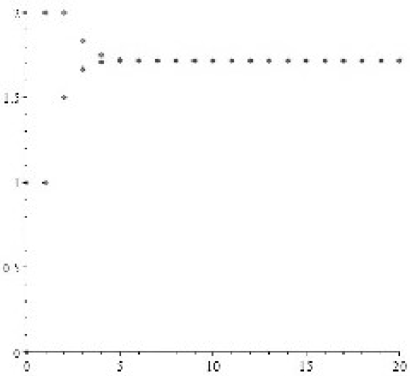
adjacentes



5.jpg

adjacent sequences

Exercise : prove that the two sequences defined by

$$\begin{cases} u_{n+1} = \sqrt{u_n v_n} \\ v_{n+1} = \dfrac{u_n + v_n}{2} \end{cases}, \text{ with } u_0 = a < v_0 = b$$

5 Subsequences

Definition 5.1 *given a sequence $(u_n)_{n\in\mathbb{N}}$, we say that $(v_n)_{n\in\mathbb{N}}$ is an extracted sequence or subsequence, if there exist an application :*

$\varphi : \mathbb{N} \to \mathbb{N}$ *strictly increasing, such that for all $n \in \mathbb{N}, v_n = u_{\varphi(n)}$.*

5.1 Property of φ :

If $\varphi : \mathbb{N} \to \mathbb{N}$ is strictly increasing application, so for all $n \in \mathbb{N}$, we have $\varphi(n) \geq n$. In particular, the sequence $(w_n)_{n\in\mathbb{N}} = \varphi(n)$ has a limit $+\infty$.

Proposition 5.1 *Let $(u_n)_{n\in\mathbb{N}}$ a sequence. If $\lim_{n\to\infty} u_n = l$, then any subsequence $(u_{\varphi(n)})_{n\in\mathbb{N}}$ also $\lim_{n\to\infty} u_{\varphi(n)} = l$.*

Corollary 5.1 *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence. If $(u_n)_{n\in\mathbb{N}}$has two subsequences converging to distinct limits then $(u_n)_{n\in\mathbb{N}}$is **divergent**, or if $(u_n)_{n\in\mathbb{N}}$has a subsequence that*

diverges *then* $(u_n)_{n\in\mathbb{N}}$ ***diverges.***

5.2 The Bolzano-Weierstrass Theorem

Theorem 5.1 *Every* ***bounded*** *sequence contains a* ***convergent subsequence.***

5.3 Limsup and Liminf

Definition 5.2 *$a \in \mathbb{R}$ is called an* ***accumulation value of*** *$(u_n)_{n\in\mathbb{N}}$ if there is* ***subsequence*** *$(v_n)_{n\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$; with $\lim v_n = a$ for $n \to \infty$*

Definition 5.3 *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers Let $Ad(u_n)_{n\in\mathbb{N}}$ be the set of* ***accumulation values*** *in $\mathbb{R}$. We called* ***limit superior*** *(resp.inferior) of $(u_n)_{n\in\mathbb{N}}$* ***is the largest accumulation value*** *(resp. the smallest) of u_n . we note*

$$\begin{aligned}\overline{\lim} u_n &= \sup Ad(u_n)_{n\in\mathbb{N}} \\ \underline{\lim} u_n &= \inf Ad(u_n)_{n\in\mathbb{N}}.\end{aligned}$$

The sequence $((-1)^n)_{n\in\mathbb{N}}$ is divergent because it has two extracted subsequences $(u_{2k})_{k\in\mathbb{N}}$ and $(u_{2k+1})_{k\in\mathbb{N}}$ converging to different limits (-1) and 1. in fact, we have

$$\begin{aligned}\lim_{k\to\infty} u_{2k} &= \lim_{k\to\infty} (-1)^{2k} = \lim_{k\to\infty} 1 = 1, \\ \lim_{k\to\infty} u_{2k+1} &= \lim_{k\to\infty} (-1)^{2k+1} = \lim_{k\to\infty} -1 = -1.\end{aligned}$$

Moreover,

$$Ad(u_n) = \{-1, 1\}$$

therefor

$$\begin{aligned}\overline{\lim} u_n &= \sup Ad(u_n) = 1 \\ \underline{\lim} u_n &= \inf Ad(u_n) = -1.\end{aligned}$$

6 Cauchy sequences

We may need to show that a sequence is convergent without necessarily explicity calculate its limit. This is the case for example when this limit is difficult to find. There is then a criterion that works well for real sequences. This is the Cauchy criterion before defining it let's start by introducing the cauchy sequences.

Definition 6.1 ***(Cauchy sequences)*** *A sequence $(u_n)_{n\in\mathbb{N}}$ of $\mathbb{R}$is called a* ***Cauchy sequence*** *if*

$$\forall\varepsilon > 0, \exists N \in \mathbb{N} \text{ sach that } \forall p, q \in \mathbb{N}, , : p \geq q \geq N \Rightarrow |u_p - u_q| < \varepsilon.$$

Proposition 6.1 ***(bounded Cauchy sequence)*** *A* ***Cauchy*** *sequence of real numbers is* ***bounded****.*

Proposition 6.2 ***(Cauchy Convergence Criterion)*** *A sequence of real numbers* ***converges*** *if and only if it is a* ***Cauchy*** *sequence.*

Example 6.1 *The geometric sequence $(u_n)_{n\in\mathbb{N}}$ defined by its general term $u_n = K^n; 0 < K < 1$ is cauchy sequence .Indeed*

Let $\forall\varepsilon > 0, \exists N \in \mathbb{N}$ sach that $\forall p, q \in \mathbb{N} : p \geq q \geq N \Rightarrow |u_p - u_q| < \varepsilon$?
We have $|u_p - u_q| = |K^p - K^q|$ ($p \geq q \Rightarrow p = q + n, n \in \mathbb{N}$) then $|u_p - u_q| = K^q |K^n - 1| \leq K^q \quad (0 < K < 1)$
Since $\exists N \in \mathbb{N},\ K^q \leq K^N < \varepsilon \Rightarrow N \ln K < \ln\varepsilon \Rightarrow N > \dfrac{\ln\varepsilon}{\ln K}$, we can take $N = E\left(\dfrac{\ln\varepsilon}{\ln K}\right) + 1$,
so for all $\varepsilon > 0; \exists N \in \mathbb{N}$ sach that $\forall p, q \in \mathbb{N} : p \geq q \geq N \Rightarrow |u_p - u_q| < \varepsilon$. which means that $(u_n)_{n\in\mathbb{N}}$ is Cauchy.

Example 6.2 *Let (u_n) be a sequence defined by $u_n = \sum_{k=2}^{n} \frac{1}{k^2}$, $k \in \mathbb{N}/\{0, 1\}$ .Show*

that (u_n) is Cauchy sequence.

We have

$$u_n = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ... + \frac{1}{n^2}.$$

Let $\varepsilon > 0$ and $p, q \in \mathbb{N}$.

$$\begin{aligned} u_p &= \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ... + \frac{1}{p^2} \\ u_q &= \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{p^2} + \frac{1}{(p+1)^2} + ... + \frac{1}{q^2}. \end{aligned}$$

If $p > q$. Then

$$\begin{aligned} |u_p - u_q| &= \left| \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{p^2} - \left(\frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{p^2} + \frac{1}{(p+1)^2} + ... + \frac{1}{q^2} \right) \right| \\ &= \left| - \left(\frac{1}{(p+1)^2} + ... + \frac{1}{q^2} \right) \right| \\ &= \frac{1}{(p+1)^2} + ... + \frac{1}{q^2}. \end{aligned}$$

Since

$$\begin{aligned} p+1 &\geq p \Longrightarrow (p+1)^2 \geq p(p+1) \\ &\Longrightarrow \frac{1}{(p+1)^2} \leq \frac{1}{p(p+1)} = \frac{1}{p} - \frac{1}{p+1}, \end{aligned}$$

$$\begin{aligned} p+2 &\geq p+1 \Longrightarrow (p+2)^2 \geq (p+1)(p+2) \\ &\Longrightarrow \frac{1}{(p+2)^2} \leq \frac{1}{(p+1)(p+2)} = \frac{1}{p+1} - \frac{1}{p+2}, \end{aligned}$$

$$\begin{aligned} q &\geq q-1 \Longrightarrow (q)^2 \geq (q-1)q \\ &\Longrightarrow \frac{1}{(q)^2} \leq \frac{1}{(q-1)q} = \frac{1}{q-1} - \frac{1}{q}. \end{aligned}$$

then,

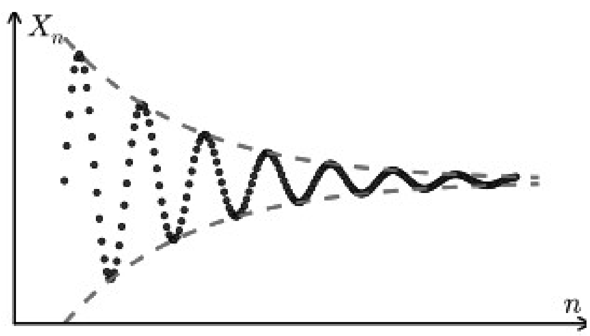
$$\begin{aligned}
|u_p - u_q| &= \frac{1}{(p+1)^2} + ... + \frac{1}{q^2} \\
&\leq \frac{1}{p} - \frac{1}{p+1} + \frac{1}{p+1} - \frac{1}{p+2} + ... + \frac{1}{q-1} - \frac{1}{q} \\
&= \frac{1}{p} - \frac{1}{q} \\
&< \frac{1}{p}.
\end{aligned}$$

Note that $\lim_{p\to\infty} \frac{1}{p} = 0$, i.e $\forall \varepsilon > 0, \exists p_0 \in \mathbb{N}$ such that $\forall p \geq p_0 \in \mathbb{N} : \frac{1}{p} < \varepsilon$. So, for $\varepsilon > 0$ which we will fix in advance, there exists a rank $n_0 = p_0 \in \mathbb{N}$ such that :

$$p \geq n_0,\, q \geq n_0 \Longrightarrow |u_p - u_q| < \frac{1}{p} < \varepsilon.$$

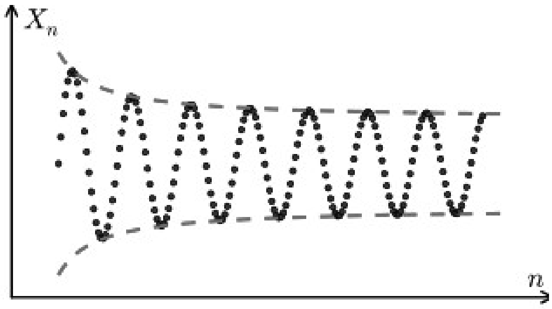
which means that (u_n) is Cauchy.

Graphic **representation of Cauchy sequence**



We say that the sequence does not cauchy if it verify

$$\exists \varepsilon > 0, \forall N \in \mathbb{N} \text{ tel que } \exists p, q \in \mathbb{N}, : p \geq q \geq N \ \wedge \ |u_p - u_q| \geq \varepsilon.$$



Example : show that $(u_n)_{n\in\mathbb{N}}$ defined by general term $u_n = \sqrt{n}$ does not a cauchy sequence

$|u_p - u_q| = \left|\sqrt{p} - \sqrt{q}\right|$;taking $p = 4(N+1)$ and $q = N+1$, we find that

$|u_p - u_q| = \left|\sqrt{4(N+1)} - \sqrt{N+1}\right| = \sqrt{N+1} \geq 1, \forall N \in \mathbb{N},$

so $\exists\varepsilon = 1 > 0$ such that $\forall N \in \mathbb{N}, \exists\ p = 4(N+1), \exists q = N+1 : p \geq q \geq N \wedge |u_p - u_q| \geq 1$

7 Recurring sequence :

Definition 7.1 *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be an application.We assume that $f(D) \subset D$.*

The sequence $(u_n)_{n\in\mathbb{N}}$ defined by the initial term $u_0 \in D$ and the recurrent relation :$\forall n \in \mathbb{N} : u_{n+1} = f(u_n)$
is called recurring sequence. This sequence is well defined because, for any integer n, we have $u_n \in D$ and $f(D) \subset D$.
•For the study of recurring sequences we need some elementary properties of continuous and differentiable maps.

7.1 General method for studying a recurring sequence

The study of the monotony of the sequence returns to that of the function f.

1) f is increasing

Proposition : Let $(u_n)_{n\in\mathbb{N}}$ be a recurring sequence, we will assume that f is **increasing**. Then the sequence $(u_n)_{n\in\mathbb{N}}$ is **monotonic**.
Furthermore, if $f(u_0) = u_1 > u_0$. So the sequence $(u_n)_{n\in\mathbb{N}}$ is increasing. If $f(u_0) = u_1 < u_0$.Then the sequence $(u_n)_{n\in\mathbb{N}}$ is decreasing.

2) f is decreasing

If f is decreasing, $u_{n+1} - u_n$ is alternately positive and negative. In this case we will consider the function $g = f \circ f$:
Let $(u_n)_{n\in\mathbb{N}}$ be a recurring sequence, we will assume that f is decreasing. Then

1) The function $g = f \circ f$ is increasing.

2) The seqences $(u_n)_{n\in\mathbb{N}}$ and $(u_{2n+1})_{n\in\mathbb{N}}$ defined by

$$\begin{aligned} u_2 &= f(f(u_0)) = g(u_0), \quad u_{2n+2} = g(u_{2n}) = f(f(u_{2n})), \; n = 1, 2, ... \\ u_1 &= f(u_0), \quad u_{2n+1} = g(u_{2n-1}) = f(f(u_{2n-1})), \; n = 1, 2, ..., \; u_0 \text{ given.} \end{aligned}$$

are monotonic and vary in opposite directions.

7.2 Fixed point theorem

Theorem 7.1 *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence defined by* $\begin{cases} u_0 \in D \\ \forall n \in \mathbb{N} : u_{n+1} = f(u_n) \end{cases}$

If (u_n) converges to a real $l \in D$ and if f is continuous in l, then we have necessarily $f(l) = l$
The real l is called a fixed point of f.

Example 7.1 *We consider the sequence $(u_n)_{n\in\mathbb{N}}$ defined by the recurrent relation as follows*

$$\begin{cases} u_0 = \frac{3}{2}, \\ \forall n \in \mathbb{N} : u_{n+1} = (u_n - 1)^2 + 1. \end{cases}$$

Let us show that $(u_n)_{n\in\mathbb{N}}$ is bounded and strictly monotone. Furthermore, deduce that $(u_n)_{n\in\mathbb{N}}$ is convergent and determine its limit.

proofove that $(P_n) : 1 < u_n < 2,$ for all $n \in \mathbb{N}$

Let us show by induction that (P_n) is true $\forall n \in \mathbb{N}$.For $n = 0$, by hypothesis we know that $1 < u_0 = \frac{3}{2} < 2$. So P_0 is true.

Suppose that the proposition (P_n) is true $\forall n \in \mathbb{N}$, and prove that (P_{n+1}) is also true $(1 < u_{n+1} < 2)$. As $1 < u_n < 2$ (recurrence hypothesis).

So $0 < u_n - 1 < 1$, hence $0 < (u_n - 1)^2 < 1$.

then $1 < u_n^2 - 2u_n^2 + 1 < 2.$

i.e (P_{n+1}) is also true.

2) Let's calculate $u_{n+1} - u_n$, we have

$$\begin{aligned} u_{n+1} - u_n &= (u_n - 1)^2 + 1 - u_n \\ &= u_n^2 - 2u_n + 1 + 1 - u_n \\ &= u_n^2 - 3u_n + 2 \\ &= (u_n - 1)(u_n - 2). \end{aligned}$$

According to the first question $0 < (u_n - 1)$ and $(u_n - 2) < 0$ so $(u_n - 1)(u_n - 2) < 0$.

then $u_{n+1} - u_n < 0$, since the sequence $(u_n)_{n\in\mathbb{N}}$ is strictly decreasing.

3) As the sequence $(u_n)_{n\in\mathbb{N}}$ is decreasing et bounded below by 1, therefore it is convergent to l. such that l verified

$l = (l-1)^2 + 1 \Longrightarrow l = l^2 - 2l + 2 \Longrightarrow l^2 - 3l + 2 = 0$. So $l = 0$ or $l = 1$. Since the first term $u_0 = \frac{3}{2}$ and $(u_n)_{n\in\mathbb{N}}$ is decreasing, then $l = 1$. **1.**

Exercise : Discus the nature of the following recurring sequence

$$\begin{cases} u_{n+1} = \sqrt{2u_n + 3} \\ u_0 \in \mathbb{R}^+ \end{cases}$$