University of Badji-Mokhtar Annaba
Faculty of Technologie
Computer Science Department

Annaba 2023-2024

Mathematical Analysis 1

Series of exercies 3: Sequences

Exercise 1: Study the nature of the following sequences and determine their possible limit:

$$B_n = \frac{\sqrt{n} - n + 1}{2\sqrt{n} + n + 2}, D_n = \frac{1! + 2! + ... + (n+1)!}{(n+1)!}, C_n = \frac{\sqrt{n+1} - \sqrt{n}}{2 + \sin\sqrt{n}}$$

Exercise 2: Using the definition of the limit of a sequence, prove that

$$\lim_{n\to+\infty} \frac{3\sqrt{n}}{4\sqrt{n}+5} = \frac{3}{4}. \quad 2) \lim_{n\to+\infty} \frac{n^2}{4n^2-1} = \frac{1}{4}. \quad 3) \lim_{n\to+\infty} \frac{(-1)^n}{2n+1} = 0,$$

4) $\lim_{n\to+\infty} \ln(n) = +\infty$.

Exercise 3 :Study monotony then convergence of sequences

$$x_n = \frac{1 \times 3 \times 5 \times ... (2n-1)}{2 \times 4 \times 6 \times ... 2n} \quad ; \bullet \; y_n = \sum_{k=1}^{n} \frac{1}{k^p} \; (p \geq 2)$$

Exercise 4: Consider the sequence $(u_n)_{n\in\mathbb{N}^*}$

$$u_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + ... + \frac{1}{2n}$$

1- Prove that $(u_n)_{n\in\mathbb{N}^*}$ is increasing.

2- Prove that $(u_n)_{n\in\mathbb{N}^*}$ is convergent and the limit l verifies $\frac{1}{2} \leq l \leq 1$

Exercise 5: Let u_0 and v_0 be two real numbers such that $0 < u_0 < v_0$.
We define two sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$

$$\text{by :} \begin{cases} u_{n+1} = \dfrac{2u_n v_n}{u_n + v_n} \\ v_{n+1} = \dfrac{u_n + v_n}{2} \end{cases}$$

1) Prove that :$\forall n \in \mathbb{N} : 0 < u_n < v_n$
2) Show that the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are adjacent.

Exercise 6: Show that the following sequences :
$U_n = \sin\left(\frac{n\pi}{2}\right), \quad V_n = \cos\left(\frac{n\pi}{2}\right), W_n = (-1)^n n$
does not converge

Exercise 7: Let the sequence $(u_n)_{n\in\mathbb{N}}$ defined by ;

$$\begin{cases} u_0 = 1 \\ u_{n+1} = \dfrac{3u_n + 2}{u_n + 2} \end{cases}$$

1) Find the constants $a, b \in \mathbb{R}$ such that $u_{n+1} = a + \frac{b}{u_n + 2}$.

2) Prove by induction that : $\forall n \in \mathbb{N}$, we have : $0 < u_n < 2$.

3) Study the monotonicity of $(u_n)_{n\in\mathbb{N}}$.

4) Show that for all $n \in \mathbb{N}$, we have : $|u_{n+1} - 2| < \frac{1}{2}|u_n - 2|$.

5) Deduce that $\forall n \in \mathbb{N}$, we have : $|u_n - 2| < \left(\frac{1}{2}\right)^n |u_0 - 2|$.

6) According to question 5, what can we conclude for the convergence of $(u_n)_{n\in\mathbb{N}}$.

Exercise 8

1- Give the mathematical definition of a sequence which is not a cauchy sequence.

2- Show that the sequence (u_n) defined by $u_n = 1 + \frac{1}{2} + ... \frac{1}{n}$ is not a cauchy sequence.

Exercices supplémentaires

Exercice 1 : Study the convergence of sequences :

$$A_n = \frac{4^{n+1} + 3^{n+1}}{4^n + 3^n}, \qquad B_n = \frac{E\left(\sqrt{n}\right)}{n}, \qquad C_n = \frac{(-1)^n \sin\left(n^p\right)}{n^p}, p \in \mathbb{N}^*$$

Exercise 2 : Consider the sequence $(u_n)_{n\in\mathbb{N}}$ defined by:

$$\begin{cases} u_0 = -1 \\ u_{n+1} = \dfrac{3 + 2u_n}{2 + u_n} \end{cases}$$

1) show that for all $n \in \mathbb{N}^*, u_n \geq 0$.

2) Show that for all $n \in \mathbb{N}^*$. $\sqrt{3} - u_{n+1} = \dfrac{\left(2 - \sqrt{3}\right)\left(\sqrt{3} - u_n\right)}{2 + u_n}$

3) Prove that for all $n \in \mathbb{N}^*, u_n \leq \sqrt{3}$

4) Study the monotony of the sequence (u_n).

5) What can we conclude?

Exercise3: Let a and b two reals such that $0 < a < b$.

We define two sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ by : $\begin{cases} u_{n+1} = \sqrt{u_n v_n} \\ v_{n+1} = \dfrac{u_n + v_n}{2} \end{cases}$

Show that $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are adjacent sequences