Chapter 4

Real-valued functions

1 Generalities on functions

Definition 1.1 *Any application from D to $\mathbb{R}$ is called a real function.If $D \subset \mathbb{R}$, we say that f is a real function of a real variable,.*

We write; $f: \begin{array}{l} D_f \rightarrow \mathbb{R} \\ x \longmapsto f(x) = y \end{array}$

D_f is called the domain of definition of . f, and $f(D_f) = \{y \in \mathbb{R}, \exists x \in D_f; \quad f(x) = y\}$

1.1 Operations on Functions

Let f and g two real funtions; $D \rightarrow \mathbb{R}$

1. $f = g \Leftrightarrow f(x) = g(x) \qquad \forall x \in D$

2. $f \leq g \Leftrightarrow f(x) \leq g(x) \qquad \forall x \in D$

If $D_f \cap D_g \neq \varnothing$ then

3. $(f+g)(x) = f(x) + g(x) \qquad \forall x \in D$

4. $(f.g)(x) = f(x).g(x) \qquad \forall x \in D$

5. $\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)} \qquad \forall x \in D_f \cap D_g$ and $g(x) \neq 0$

6.Let $f : D_f \rightarrow \mathbb{R}$ and $g : D_g \rightarrow \mathbb{R}$ with $f(D_f) \subset D_g$. We define the composite function $(g \circ f)$ in D_f : $\quad (g \circ f)(x) = g(f(x))$

Example $g(x) = \cos x, f(x) = \dfrac{1}{x} \Rightarrow (g \circ f)(x) = g\left(\dfrac{1}{x}\right) = \cos\left(\dfrac{1}{x}\right)$

1.2 Properties of functions :

Even and odd functions

A set $D \subset \mathbb{R}$ is said to be symmetric with respect to the origin if for all $x \in D \Rightarrow -x \in D$

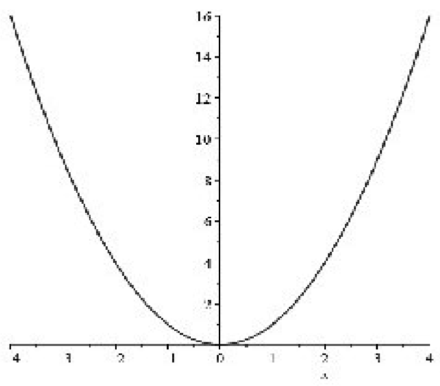
Definition 1.2 *A function f defined on the symmetric set D is said to be*

i)Even if $\quad \forall x \in D, \; f(x) = f(-x)$

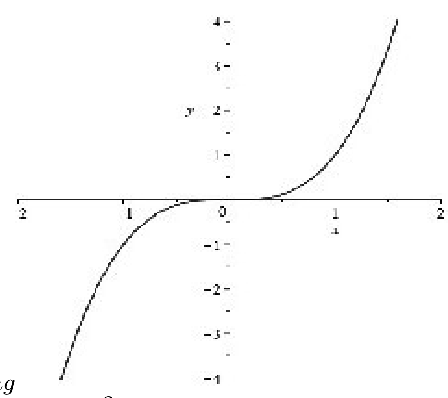
ii) Odd if $\quad \forall x \in D, \; f(-x) == -f(x)$

Examples : i) $f(x) = x^2$; $\quad f(x) = \cos x$; $\; f(x) = |x|$

ii) $f(x) = x^3$; $\quad f(x) = \sin x$;



X^3



2.png

x^2 is even $\qquad$ x^3 is odd

When f is even, the graph of the function f is symmetric with respect to the y-axis. If f is odd, the graph of f is symmetric with respect to the origin.

Periodic function

Let $f : \; D \rightarrow \mathbb{R}$ is said periodic if $\exists \; \alpha \in \mathbb{R}$ such that

i) $x + \alpha \in D$

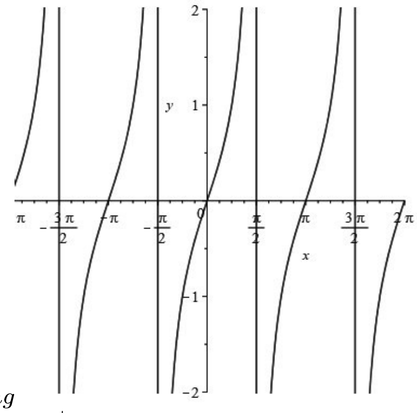
ii) $f(x+\alpha) = f(x) \quad \forall x \in D$

It is obvious that $\quad f(x + k\alpha) = f(x) \quad \forall k \in \mathbb{N}^*$

Definition 1.3 *We call the period of f the smallest positive number T such that* $f(x+T) = f(x) \ \ \forall x \in D$

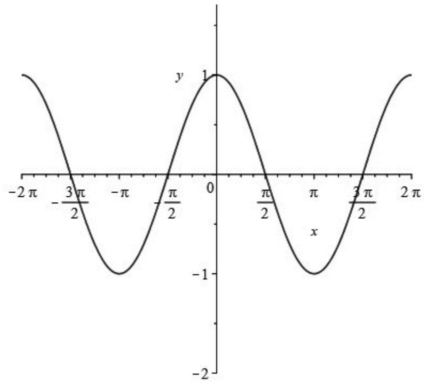
Example : $f(x) = \sin x; \quad f(x) = \cos x; \ \ f(x) = \tan x$

tanx



x

cos



3.png $\tan x$ 4.png $\cos x$

Monotonic Function

Let $f: \ D \rightarrow \mathbb{R}$, be a real function

a) f is increasing on D .if $\forall x_1, x_2 \in D, x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$

b) Strictly increasing si $\forall x_1, x_2 \in D, x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

c) f is decreasing if $\forall x_1, x_2 \in D, x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$

d) Strictly decreasing $\forall x_1, x_2 \in D, x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Example :

› The exp : $\mathbb{R} \rightarrow \mathbb{R}$ and $\ln :]0, +\infty[\rightarrow \mathbb{R}$ are Strictly increasing.

› $|x| : \mathbb{R} \rightarrow \mathbb{R}$ not increasing and not decreasing. On other hand , the function $|x| : [0, +\infty[\rightarrow \mathbb{R}$ is strictly increasing..

Bounded Functions

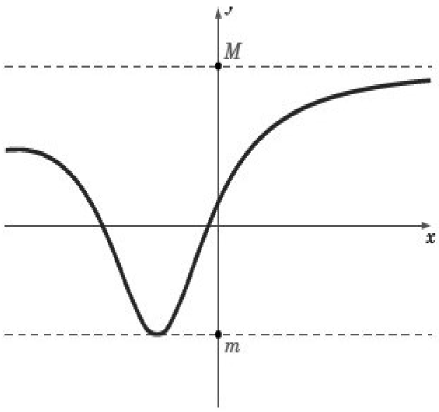
The function f is called :

I) If there exists $M \in \mathbb{R}$ such that : $f(x) \leq M, \forall x \in D_f$, ,then the function f is said to be bounded above by M.

II) If there exists $m \in \mathbb{R}$ such that : $f(x) \geq m, \forall x \in D_f$, then the function f is said to be bounded below by m

III) f is bouned on D_f if $\begin{cases} \exists m, M \in \mathbb{R} : m \leq f(x) \leq M, \forall x \in D & \text{or} \\ \exists c \in \mathbb{R}_+ : |f(x)| \leq c, \forall x \in D \end{cases}$

born



5.*png*

We will call, if it exists, the upper bound (respectively lower bound) of the function f on the domain D_f

Example 1.1 › *The function* $\begin{matrix} f : [0,1] \to \mathbb{R} \\ x \mapsto x^2 - 3 \end{matrix}$ *is bounded on* $[0,1]$

with $-3 \leq f(x) \leq -2$

›*the function* $\cos x$ *is bounded on* $\mathbb{R}$ *because , we have* $\forall x \in \mathbb{R} : |\cos x| \leq 1$

Remark 1.1 *as we saw in chapter 1, it is possible to write*

$$M = \sup_D f \Leftrightarrow \begin{cases} \forall x \in D : f(x) \leq M \\ \forall \varepsilon > 0, \exists x \in D : f(x) > M - \varepsilon \end{cases}$$

Injective, surjective, and bijective functions

Let $f : D \to \mathbb{R}$ be a real function

- f is injective if the only if $\forall x, x' \in D;\ f(x) = f(x') \Rightarrow x = x'$

- f is surjective if, $\forall y \in \mathbb{R}, \exists x \in D$ such that $f(x) = y$
- f is bijective if f is injective and surjective

2 Limits

The essence of the concept of limit for real-valued functions of a real variable is this : If L is a real number, then $\lim_{x\to x_0} f(x) = L$ means that the value $f(x)$ can be made as close to L as we wish by taking x sufficiently close to x_0. This is made precise in the following definition.

Definition 2.1 *We say that $f(x)$ approaches the limit L as x approaches x_0, and write* $\lim_{x\to x_0} f(x) = L$

if f is defined on some deleted neighborhood of x_0 and, for every $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - L| < \varepsilon$, if $|x - x_0| < \delta$
So

$$\lim_{x\to x_0} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in D_f : |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

A function can have a limit when x tends to x_0 without it being defined at this point.

Example 2.1 *For the function f defined by $f(x) = 2x + 3$, we have $\lim_{x\to 1} f(x) = 5$. Indeed, noting that for any $x \in \mathbb{R}$:*

$$|f(x) - l| = |2x + 3 - 5| = 2|x - 1| < \varepsilon$$

it suffices then to take $\delta = \frac{\varepsilon}{2}$ *to have* $\forall \varepsilon > 0, |x - 1| < \delta = \frac{\varepsilon}{2} \Rightarrow 2|x - 1| < \varepsilon \Rightarrow |f(x) - 5| < \varepsilon$

Example 2.2 ▷*For the function f defined by $f(x) = x^3$, we have $\lim_{x\to 0} f(x) = 0$. Indeed, noting that for any $x \in \mathbb{R} : |f(x) - l| = |x^3| < \varepsilon$.*

it suffices then to take $\delta = \sqrt[3]{\varepsilon}$ to have $\forall \varepsilon > 0, |x - 0| < \delta \Rightarrow |x| < \sqrt[3]{\varepsilon} \Rightarrow |f(x) - l| < \varepsilon$

Example 2.3 ▷*Show that* $\lim_{x\to -2} \frac{x}{1-x} = \frac{-2}{3}$

by definition : $\lim\limits_{x\to -2}\dfrac{x}{1-x}=\dfrac{-2}{3}\Leftrightarrow \forall\varepsilon>0,\exists\delta>0,\forall x\in D_f:|x+2|<\delta\Rightarrow \left|\dfrac{x}{1-x}+\dfrac{2}{3}\right|<\varepsilon$

We have $\left|\dfrac{x}{1-x}+\dfrac{2}{3}\right|=\left|\dfrac{x+2}{3(1-x)}\right|=\dfrac{|x+2|}{3|1-x|}$

let's choose a neighborhood of $x_0=-2;|x+2|<1....(*)\Leftrightarrow -1<x+2<1$

$\Rightarrow -3<x<-1\Leftrightarrow 1<-x<3\Rightarrow 2<1-x<4\Leftrightarrow \dfrac{1}{4}<\dfrac{1}{1-x}<\dfrac{1}{2}\Leftrightarrow \dfrac{1}{|1-x|}<\dfrac{1}{2}$

so $\dfrac{|x+2|}{3|1-x|}<\dfrac{|x+2|}{3.2}=\dfrac{|x+2|}{6}<\varepsilon$ then , we obtain $|x+2|<6\varepsilon...(**)$

for $(*)$ and $(**)$, we take $\delta=\min(1,6\varepsilon)$

Theorem 2.1 *If $\lim\limits_{x\to x_0}f(x)$ exists; then it is unique I that is; if $\lim\limits_{x\to x_0}f(x)=L_1$ and $\lim\limits_{x\to x_0}f(x)=L_2$ then $L_1=L_2$*

Theorem 2.2 *Let $f:D\to\mathbb{R}$ be a real function; Therefore, we have the equivalence of the following propositions*

▲ *i)* $\lim\limits_{x\to x_0}f(x)=l.$

▲▲ *ii) For all sequence $(x_n)_{n\in\mathbb{N}}$ on D which converge to x_0 ($x_n\neq x_0$) then the sequence $f(x_n)$ converge to $f(x_0)$ on $f(D)$*

Example 2.4 *$f:\mathbb{R}^*\to\mathbb{R}$ such that $f(x)=\sin\dfrac{1}{x}$*

we show that f not admit the limit for $x\to 0$

Let a sequence $u_n=\dfrac{1}{\pi\left(n+\frac{1}{2}\right)}$. itis clair that $\lim\limits_{n\to+\infty}u_n=0$ and $\forall n\in\mathbb{N};u_n\neq 0$

$$f(u_n)=\sin\pi\left(n+\frac{1}{2}\right)=(-1)^n=\begin{cases}1 \text{ if } n \text{ is even}\\ -1 \text{ if } n \text{ os odd}\end{cases}.$$

Then $\lim\limits_{n\to+\infty}f(u_n)=\pm 1$ so $f(u_n)$ is a divergente sequence , therefore f not admit a limit at x_0

Definition 2.2 *Given a function $f:D\to\mathbb{R}$. We say that f ,has a right-hand limit at the point x_0 if and only if : $\lim\limits_{x\overset{>}{\to}x_0}f(x)=l\Leftrightarrow\forall\varepsilon>0,\exists\delta_\varepsilon>0,\forall x\in D:0<x-x_0<$*

$\delta_\varepsilon \Rightarrow |f(x) - l| < \varepsilon.$

Definition 2.3 *We say that f ,has a left-hand limit at the point x_0 if and only if :* $\lim\limits_{x \overset{<}{\to} x_0} f(x) = l \Leftrightarrow \forall \varepsilon > 0, \exists \delta_\varepsilon > 0, \forall x \in D : 0 < x_0 - x < \delta_\varepsilon \Rightarrow |f(x) - l| < \varepsilon.$

<u>**Notation**</u> : We note that $0 < x - x_0 < \delta_\varepsilon \Leftrightarrow x \in]x_0, x_0 + \delta_\varepsilon[$ and $0 < x_0 - x < \delta_\varepsilon \Leftrightarrow x \in]x_0 - \delta_\varepsilon, x_0[$.

Theorem 2.3 *We have the following equivalence :* $\lim\limits_{x \to x_0} f(x) = l \Leftrightarrow \lim\limits_{x \overset{>}{\to} x_0} f(x) = l = \lim\limits_{x \overset{<}{\to} x_0} f(x)$

Consequence : if $\lim\limits_{x \overset{>}{\to} x_0} f(x) \neq \lim\limits_{x \overset{<}{\to} x_0} f(x)$ then $\lim\limits_{x \to x_0} f(x)$ does not exist

Example 2.5 *we consider the function* $f(x) = \dfrac{|x(x+1)|}{x+1} = \begin{cases} \dfrac{x(x+1)}{x+1} & \text{if } x \in]-\infty, -1[\cup [0, +\infty \\ -\dfrac{x(x+1)}{x+1} & \text{if } x \in]-1, 0[\end{cases}$

,we have $\begin{cases} \lim\limits_{x \overset{>}{\to} -1} f(x) = \lim\limits_{x \overset{>}{\to} -1} x = -1 \\ \lim\limits_{x \overset{<}{\to} -1} f(x) = \lim\limits_{x \overset{<}{\to} -1} -x = 1 \end{cases} \Rightarrow \lim\limits_{x \overset{>}{\to} -1} f(x) \neq \lim\limits_{x \overset{<}{\to} -1} f(x)$ *then* $\lim\limits_{x \to -1} f(x)$ *does not exist*

2.1 Infinite limits

Definition 2.4 *Let f be a function defined on a domain $D \subset \mathbb{R}$. We say that f tends to $+\infty$ $(resp\ -\infty)$ as x tends to x_0 if and only if : :*

$$\lim_{x \to x_0} f(x) = +\infty \Leftrightarrow \forall A > 0, \exists \delta > 0, \forall x \in D : |x - x_0| < \delta \Rightarrow f(x) > A$$

respectively

$$\lim_{x \to x_0} f(x) = -\infty \Leftrightarrow \forall A > 0, \exists \delta > 0, \forall x \in D : |x - x_0| < \delta \Rightarrow f(x) < -A.$$

Example 2.6 *Using the definition of the limit, show that :* $\lim\limits_{\substack{> \\ x\to 0}} \ln x = -\infty$

By definition, we have $\lim\limits_{\substack{> \\ x\to 0}} \ln x = -\infty \Leftrightarrow \forall A > 0, \exists ?\delta > 0, \forall x > 0 : |x| < \delta \Rightarrow \ln x < -A$

$\ln x < -A \Leftrightarrow x < e^{-A}$, *so for any* $A > 0$, *we can choose* $\delta = e^{-A}$ *to ensure that the inequality* $|x| < \delta \Rightarrow \ln x < -A$

2.2 Finite limits of functions as x tends to $\pm\infty$

Definition 2.5 *Let f be a function defined on a domain* $D \subseteq \mathbb{R}$. *We say that* f *tends to* L *as* x *tends to* $+\infty$ *if and only if : :*

$$\lim_{x\to+\infty} f(x) = L \Leftrightarrow \forall\varepsilon > 0, \exists A > 0, \forall x \in D : x \geq A \Rightarrow |f(x) - L| < \varepsilon$$

respectively

$$\lim_{x\to-\infty} f(x) = l \Leftrightarrow \forall\varepsilon > 0, \exists B > 0, \forall x \in D : x \leq -B \Rightarrow |f(x) - l| < \varepsilon$$

2.3 Infinite limits of functions as x tens to $\pm\infty$

Definition 2.6 *Let f be a function defined on an interval of type* $[a, +\infty[\subset D_f$.

• *We say that* f *tends to* $+\infty$ *as* x *tends to* $+\infty$ *iff :*

$$\lim_{x\to+-\infty} f(x) = +\infty \Leftrightarrow \forall A > 0, \exists B > 0, \forall x \in D : x \geq B \Rightarrow f(x) \geq A$$

• *We say* f *tends to* $+\infty$ *as* x *tend to* $-\infty$ *iff :*

$$\lim_{x\to+-\infty} f(x) = -\infty \Leftrightarrow \forall A > 0, \exists B > 0, \forall x \in D : x \geq B \Rightarrow f(x) \leq -A$$

Definition 2.7 *Let f be a function defined on an interval of type* $]-\infty, a] \subset D_f$.

• f *tends to* $+\infty$ *as* x *tends to* $-\infty$ *iff :*

$$\lim_{x\to-\infty} f(x) = +\infty \Leftrightarrow \forall A > 0, \exists B > 0, \forall x \in D : x \leq -B \Rightarrow f(x) \geq A$$

- *f tends to $-\infty$ as x tends to $-\infty$ iff :*

$$\lim_{x\to--\infty} f(x) = -\infty \Leftrightarrow \forall A > 0, \exists B > 0, \forall x \in D : x \leq -B \Rightarrow f(x) \leq -A$$

2.4 Algebraic operations on limits and some properties

Let f and g be two functions definied,on the sameset $D \subseteq \mathbb{R}$,such that $\lim\limits_{x\to x_0} f(x) = l$ and $\lim\limits_{x\to x_0} g(x) = l'$
then we have :

- $\lim\limits_{x\to x_0} (f \pm g)(x) = l \pm l'$
- $\lim\limits_{x\to x_0} (f.g)(x) = l.l'$
- $\lim\limits_{x\to x_0} f^n(x) = l^n$
- $\lim\limits_{x\to x_0} |f(x)| = |l|$
- $\lim\limits_{x\to x_0} (\lambda f + \mu g)(x) = \lambda l + \mu l'$
- $\lim\limits_{x\to x_0} \dfrac{f}{g}(x) = \frac{l}{l'} \quad l' \neq 0$

Theorem 2.4 ***d'encadrement*** *: Let f, g, h be three functions verify $f(x) < g(x) < h(x)$ in the neighborhood of x_0*

If $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} h(x) = l$ then $\lim\limits_{x\to x_0} g(x) = l$

Proposition 2.1 *1• Let f, g be two functions verify $f(x) < g(x)$ in the neighborhood of x_0*

If $\begin{cases} \lim\limits_{x\to x_0} f(x) = +\infty \ \textit{then} \ \lim\limits_{x\to x_0} g(x) = +\infty \\ \lim\limits_{x\to x_0} g(x) = -\infty \ \textit{then} \ \lim\limits_{x\to x_0} f(x) = -\infty \end{cases}$

2• Let f be a function bounded in the neighborhood of x_0 and g a function verify $\lim\limits_{x\to x_0} g(x) = 0$ then $\lim\limits_{x\to x_0} f(x)\, g(x) = 0$.

Definition 2.8 ***Important :*** *Let f :$D \to \mathbb{R}$ be a function of real variable. We say that f is defined in the neighborhood of x_0 iff : there exists an interval $I =]x_0 - \varepsilon, x_0 + \varepsilon[$ such that $I \subset D$*

2.5 Comparisons of functions in the neighborhood of point (near a point) :

Let f and g :$D \to \mathbb{R}$, and let $x_0 \in D$