Solution

Exercise 1

a) we will show by induction that $P(n);\forall n\in\mathbb{N},(1+\alpha)^n\geq 1+n\alpha$

for $n=0$, we have $(1+\alpha)=1\geq 1+0\alpha=1$,so $P(0)$ is true

assume that $P(n)$ is true and show that $P(n+1)$ is true , we show that $(1+\alpha)^{n+1}\geq 1+(n+1)\alpha$

since $1+\alpha\geq 0$ then $(1+\alpha)^{n+1}=(1+\alpha)(1+\alpha)^n\geq(1+\alpha)(1+n\alpha)=1+\alpha+n\alpha+n^2\alpha\geq 1+(n+1)\alpha \quad / \quad (n^2\alpha\geq 0)$

therefore $P(n+1)$ is true , hence

$$\forall n\in\mathbb{N},(1+\alpha)^n\geq 1+n\alpha$$

• If we put $\alpha=1$ we obtain the following inequality $\forall n\in\mathbb{N},2^n\geq n \quad (1+1)^n\geq 1+n\geq n$

b) Let $(a,b)\in\mathbb{Q}^+\times\mathbb{Q}^+$ such that $\sqrt{ab}\notin\mathbb{Q}$'

Suppose that $\sqrt{a}+3\sqrt{b}\in\mathbb{Q}$. So $\exists(p,q)\in\mathbb{Z}\times\mathbb{Z}^*$ such that $\sqrt{a}+3\sqrt{b}=\dfrac{p}{q}$

Then, $\left(\sqrt{a}+3\sqrt{b}\right)^2=\left(\dfrac{p}{q}\right)^2$ which impplies that $a+9b+6\sqrt{ab}=\frac{p^2}{q^2}$

so $\sqrt{ab}=\dfrac{\dfrac{p^2}{q^2}-a-9b}{6}\in\mathbb{Q}$ which contradicts the fact that $\sqrt{ab}\notin\mathbb{Q}$. then $\sqrt{a}+3\sqrt{b}\notin\mathbb{Q}$

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Exercise 2: integer part

$E(x)\overset{def}{\Rightarrow}$ is the greatest integer less than or equal to $x:\forall x\in\mathbb{R}$ $E(x)\leq x<E(x)+1$ or $x-1<E(x)\leq x$

1) $x=0.79\Rightarrow E(0.79)=0,E(1.08)=1,E\left(\frac{11}{3}\right)=3,E(e)=2$

2) Show that: $\forall x\in\mathbb{Z},\;\; E(x)+E(-x)=0$

• if $x\in\mathbb{Z}:\quad E(x)=x$ and $E(-x)=-x$ so $E(x)+E(-x)=0$

• if $x\in\mathbb{R}/\mathbb{Z}$ then $E(x)\leq x<E(x)+1\Rightarrow -E(x)-1<x\leq -E(x)\Rightarrow$

$-\underbrace{E(x)-1}_{K}<x\leq -\underbrace{E(x)-1}_{K}+1,\quad K\in\mathbb{Z}$ then $-E(x)-1$ is the integer part of $-x$

2) $\forall x \in \mathbb{R}$, we show that $E(\frac{x}{2})+E(\frac{x+1}{2})=E(x)$

we distinguish 2 cases $x \in \mathbb{R} \Rightarrow \begin{cases} E(x) \text{ is even} \Rightarrow E(x)=2k \\ \text{or} \\ E(x) \text{ is odd} \Rightarrow E(x)=2k+1 \end{cases}$

<u>1^{st} case</u> • $E(x)=2k \Rightarrow \quad E(x) \leq x < E(x)+1 \Rightarrow 2k \leq x < 2k+1$, we divide by 2

$k \leq \frac{x}{2} < \frac{2k+1}{2} \Rightarrow k \leq \frac{x}{2} < k+\frac{1}{2} < k+1$ from where $E\left(\frac{x}{2}\right)=k$

and $E(x)+1 \leq x+1 < E(x)+2 \Rightarrow \frac{E(x)+1}{2} \leq \frac{x+1}{2} < \frac{E(x)+2}{2}$ $(E(x)=2k)$ then

$$\frac{2k+1}{2} \leq \frac{x+1}{2} < \frac{2k+2}{2}$$

which give $k < k+\frac{1}{2} \leq \frac{x+1}{2} < k+1$

then $E(\frac{x+1}{2})=k$ d'ou $E(\frac{x}{2})+E(\frac{x+1}{2})=k+k=2k=E(x)$

<u>2^{nd} case</u> •$E(x)=2k+1$ in the same way

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<u>Exercise3</u>

1• we prove the second triangle inequality

$\forall x,y \in \mathbb{R}, \quad ||x|-|y|| \leq |x+y|$

let

$|x|=|(x+y)+(-y)| \leq |x+y|+|-y|=|x+y|+|y|$

we have

$|x|-|y| \leq |x+y| \ldots\ldots (1)$

in the same way

$|y|=|(y+x)+(-x)| \leq |x+y|+|-x|=|x+y|+|x|$

then

$|y|-|x| \leq |x+y| \Rightarrow -(|x|-|y|) \leq |x+y| \Rightarrow |x|-|y| \geq -|x+y| \ldots\ldots (2)$.

From (1) and (2) , we have $-|x+y| \leq |x|-|y| \leq |x+y|$

which implies $\boxed{||x|-|y|| \leq |x+y|}$

in the same way, we can show that $\quad ||x|-|y|| \leq |x-y|$

2• $\forall x,y \in \mathbb{R}$,we have $\quad 2|x| = |(x+y)+(x-y)| \Rightarrow 2|x| \leq |x+y|+|x-y|$

$$2|y| = |(x+y)+(y-x)| \Rightarrow 2|y| \leq |x+y| + |y-x|$$

then

$$|x| + |y| \leq |x+y| + |y-x|, \text{ for all } x, y \in \mathbb{R}$$

3.Prove that $\forall (x,y) \in \mathbb{R}^2 : 1+|xy-1| \leq (1+|x-1|)(1+|y-1|)$

Let $x, y \in \mathbb{R} : (1+|x-1|)(1+|y-1|) = 1+|x-1|+|y-1|+|(x-1)(y-1)|$

and $1+|x-1|+|y-1|+|(x-1)(y-1)| \geq 1+|x+y-2|+|xy-x-y+1|$

$\geq 1+|x+y-2+xy-x-y+1| \geq 1+|xy-1|$

then $\forall (x,y) \in \mathbb{R}^2 : 1+|xy-1| \leq (1+|x-1|)(1+|y-1|)$

Exercise 5

It holds that $\inf(A) \leq x$ for all $x \in A$ and therefore, $-\inf(A) \geq -x$ for all $x \in A$,

which is equivalent to $-\inf(A) \geq y$ for all $y \in -A$.

Therefore, $-\inf(A)$ is an **upper bound** of the set $-A$ and therefore $\sup(-A) \leq -i\inf(A)$.

Now,$y \leq \sup(-A)$ for all $y \in -A$, or equivalently $-x \leq \sup(-A)$ for all $x \in A$.

Hence,$x \geq \sup(-A)$ for all $x \in A$. This proves that $-\sup(-A)$ is a **lower bound** of A and therefore $-\sup(-A) \leq \inf(A)$, or $\sup(-A) \geq -\inf(A)$.

This proves that $\sup(-A) = -\inf(A)$.

Exercise 6:

$\bullet A = \{x \in \mathbb{R} : |2x+1| \leq 5\} = \{x \in \mathbb{R} : -5 \leq 2x+1 \leq 5\} = \left\{x \in \mathbb{R} : \frac{-6}{2} \leq x \leq \frac{4}{2}\right\}$

$= \{x \in \mathbb{R} : -3 \leq x \leq 2\} = [-3, 2]$

A is bounded with $\sup A = \max A = 2$ and $\inf A = \min A = -3$.

$\bullet B = \{x \in \mathbb{R}, e^x < \frac{1}{2}\} = \{x \in \mathbb{R}, x < \ln\frac{1}{2}\} = \{x \in \mathbb{R}, x < -\ln 2\} =]-\infty, -\ln 2[$

B is bounded above such that $\sup B = -\ln 2$, on the other hand B is not bounded below

so $\nexists \inf B$,then B is not bounded.

$\bullet C = \left\{E\left(\frac{1}{n}\right), n \in \mathbb{N}^*\right\}$

si $n = 1, E(1) = 1 \in C \Rightarrow C \neq \varnothing \Rightarrow D$ is bounded then $\exists \sup C, \exists \inf C$ such that $\forall c \in C;$ $\inf C \leq c \leq \sup C$

We have if $n > 1$ then $0 < \dfrac{1}{n} < 1$ and $E\left(\dfrac{1}{n}\right) = 0$ so $C = \{0, 1\} \Rightarrow$ $\sup C = \max C = 1$ and $\inf C = \min C = 0$

Exercise7

$\bullet A = \{\dfrac{n-2}{n+1}; n \in \mathbb{N}^*\}$

A is note empty : for $n = 0, \dfrac{n-2}{n+1} = -2 \in A$

we have $\forall n \in \mathbb{N};$ $\dfrac{n-2}{n+1} = 1 - \dfrac{3}{n+1}.$

$\forall n \in \mathbb{N},\ n+1 > 1 \Rightarrow 0 < \dfrac{3}{n+1} < 3 \Rightarrow$ $1 > 1 - \dfrac{3}{n+1} > -2.$

Then A is bounded below by -2 and bounded above by 1 then A is bounded

$\Rightarrow \exists \sup A. \inf A$ such that $\forall x_n \in A; \inf A \leq x_n \leq \sup A.$

The set of upper bound of A is $[1, +\infty[$ and the set of lower bound of A is $]-\infty, -2]$

since $-2 \in A$ then $-2 = \min A = \inf A$

$1 \notin A$, then $\nexists \max A$, show that $\sup A$=1 ,

using characterization of the Supremum

$\sup A = 1 \Leftrightarrow \begin{cases} \forall x_n \in A, x_n < 1 \\ \forall \varepsilon > 0, \exists x_{n_0} \in A \text{ such that } x_{n_0} > 1 - \varepsilon \end{cases}$ $, M = 1$

so

$x_{n_0} \in A \Rightarrow \exists n_0 \in \mathbb{N}^*$ such that $x_{n_0} = 1 - \dfrac{3}{n_0+1},$

which implies $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}^*$ such that $1 - \dfrac{3}{n_0+1} > 1 - \varepsilon \Rightarrow$ $\dfrac{3}{n_0+1} < \varepsilon$

$$\forall \varepsilon > 0, (n_0 + 1)\,\varepsilon > 3$$

According to Archimedean property : $\exists n_0 \in \mathbb{N}^*$ such that $n_0 > \dfrac{3}{\varepsilon} - 1$,just take: $n_0 = E\left(\dfrac{3}{\varepsilon} - 1\right) + 1,$

2) Let $B = \{x^2 + 1; x \in]1, 2]\}$

we have $x \in]1, 2] \Rightarrow 1 < x \leq 2 \Rightarrow 1 < x^2 \leq 4$ (x^2 increasing function)

$1+1 < x^2+1 \leq 4+1$ that s to say $\forall y \in B, y = x^2+1, 2 < y \leq 5$

then the set B is bounded below by 2 and bounded above by 5 hence B is bounded.

indeed there exist $\inf B$ and $\sup B$ such that $\forall y \in B; \inf B < y \leq \sup B$

on the other hand $5 = 2^2 + 1 \in B$, therefore $\sup B = \max B = 5$,let us show that $\inf B = 2$.

We use the characterization of infimum

$$\inf B = 2 \Leftrightarrow \begin{cases} \forall y \in B, 2 < y \\ \forall \varepsilon > 0, \exists y_0 \in B \text{ such that } 2 + \varepsilon > y_0 \end{cases}, m = 2$$

$\inf B = 2 \Leftrightarrow \forall \varepsilon > 0, \exists x_0 \in \left]1, 2\right]$ such that $2 + \varepsilon > x_0^2 + 1 \Rightarrow 1 + \varepsilon > x_0^2$

which give $\quad |x_0| < \sqrt{1+\varepsilon} \Leftrightarrow -\sqrt{1+\varepsilon} < x_0 < \sqrt{1+\varepsilon}$

so, we search for the existence of $x_0 \in \left]1, 2\right] \cap \left]-\sqrt{1+\varepsilon}, \sqrt{1+\varepsilon}\right[$,as $\varepsilon > 0$ then$\sqrt{1+\varepsilon} > 1$

hence, we distinguish two cases

•<u>If $\sqrt{1+\varepsilon} < 2$</u>

in this case we take $x_0 \in \left]1, \sqrt{1+\varepsilon}\right[$

• <u>If $\sqrt{1+\varepsilon} > 2$</u>

in this case we take any value of $x_0 \in \left]1, 2\right]$

then $\forall \varepsilon > 0, \exists x_0 \in \left]1, 2\right]$ such that $2 + \varepsilon > x_0^2 + 1$ hence $\inf B = 2, \mathit{and}\ 2 \notin B$ then $\nexists \min B$