Solution

Exercise 1

•To put a quotient in algebraic form, you must multiply the numerator and the denominator by the conjugate quantity denominator

The conjugate of $1 - i\sqrt{3}$ is $1 + i\sqrt{3}$ then

• $z_1 = \dfrac{2}{1 - i\sqrt{3}} = \dfrac{2\left(1 + i\sqrt{3}\right)}{\left(1 - i\sqrt{3}\right)\left(1 + i\sqrt{3}\right)} = \dfrac{2\left(1 + i\sqrt{3}\right)}{1+3} = \boxed{\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}}$

• $z_2 = \dfrac{i-4}{-3+2i} = \dfrac{4-i}{3-2i} = \dfrac{(4-i)(3+2i)}{(3-2i)(3+2i)}$

$= \dfrac{12+2+i(8-3)}{13} = \dfrac{14}{13} + i\dfrac{5}{13}$

• $z_3 = (1-i)^3$ using the binômial formula $\Rightarrow$

$z_3 = (1-i)^3 = 1 - 3i + 3i^2 - i^3 = \boxed{-2-2i}$

• $z_4 = \dfrac{2\exp\left(i\frac{\pi}{6}\right)}{\exp\left(i\frac{\pi}{3}\right)} = 2\exp\left(i\frac{\pi}{6}\right) \times \exp\left(-i\frac{\pi}{3}\right) = 2\exp\left(i\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\right)$

$= 2\exp\left(-i\frac{\pi}{6}\right) = 2\cos\left(-\frac{\pi}{6}\right) + 2i\sin\left(-\frac{\pi}{6}\right) = 2\dfrac{\sqrt{3}}{2} - 2\dfrac{1}{2}i = \boxed{\sqrt{3} - i}$

• z_5, the module 2 and argument $\frac{\pi}{3}$

$\Rightarrow z_5 = 2\exp\left(i\frac{\pi}{3}\right) = \boxed{1 + i\sqrt{3}}$

Exercise 2

Let's calculate the module and argument of :

$v = \dfrac{\sqrt{6} - i\sqrt{2}}{2}$ and $w = 1 - i$.

• $v = \dfrac{\sqrt{6} - i\sqrt{2}}{2} = \sqrt{2}\left(\dfrac{\sqrt{3}}{2} - i\dfrac{1}{2}\right) \Rightarrow$

$|v| = \sqrt{2}\sqrt{\left(\dfrac{\sqrt{3}}{2}\right)^2 + \left(\dfrac{1}{2}\right)^2} = \sqrt{2}$ and

$$\begin{cases} \cos\theta = \dfrac{\sqrt{3}}{2} \\ \sin\theta = -\dfrac{1}{2} \end{cases} \Rightarrow \theta = \arg v = -\dfrac{\pi}{6} + 2k\pi$$

• $u = 1 - i = \sqrt{2}\left(\dfrac{\sqrt{2}}{2} - i\dfrac{\sqrt{2}}{2}\right) \Rightarrow |u| = \sqrt{2}$ and $\arg u = -\dfrac{\pi}{4} + 2k\pi$

• $\left|\dfrac{v}{u}\right| = \dfrac{|v|}{|u|} = \dfrac{\sqrt{2}}{\sqrt{2}} = 1$ and

$\arg\dfrac{v}{u} = \arg v - \arg u = -\dfrac{\pi}{6} + \dfrac{\pi}{4} = \dfrac{\pi}{12} + 2k\pi$

Exercise 3

We look the complex numbers Z such that

$$Z^2 = \frac{\sqrt{3}+i}{2} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$$

we put $Z = x+iy \Leftrightarrow (x+iy)^2 = \frac{\sqrt{3}}{2} + i\frac{1}{2}$,
is equivalent to the system

$$\begin{cases} x^2 - y^2 = \frac{\sqrt{3}}{2} \text{..........} (1) \\ 2xy = \frac{1}{2} \text{ ....} (2) \qquad xy > 0 \Leftrightarrow \ x, y \text{ have the same sign} \\ x^2 + y^2 = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1\text{.....} (3) \end{cases}$$

of $(1)+(3)$ and $(3)-(1)$, we find that $x = \pm\frac{\sqrt{2+\sqrt{3}}}{2}$ and $y = \pm\frac{\sqrt{2-\sqrt{3}}}{2}$

since x and y have the same sign, we obtain $\begin{cases} Z_1 = \frac{\sqrt{2+\sqrt{3}}}{2} + i\frac{\sqrt{2-\sqrt{3}}}{2} \\ Z_2 = -\frac{\sqrt{2+\sqrt{3}}}{2} - i\frac{\sqrt{2-\sqrt{3}}}{2} \end{cases}$

On the other hind $Z^2 = \frac{\sqrt{3}}{2} + i\frac{1}{2} \overset{\text{exponential form}}{\Rightarrow}$

$$Z^2 = (r\exp(i\theta))^2 = r^2\exp(i2\theta) = \exp\left(i\frac{\pi}{6}\right)$$

by identification, we obtain $\begin{cases} r^2 = 1 \\ 2\theta = \frac{\pi}{6} + 2k\pi, k = 0, 1, ... \end{cases} \Leftrightarrow \begin{cases} r = 1 \\ \theta = \frac{\pi}{12} + k\pi, k = 0, 1, ... \end{cases}$

for $\bullet\ k = 0$;

$$Z_0 = \exp\left(i\frac{\pi}{12}\right) \overset{\text{trigonometric form}}{=} \cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)$$

$\bullet\ k = 1$; $\quad Z_1 = \exp\left(i\frac{13\pi}{12}\right) = \cos\left(\frac{13\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right)$

since $\cos\left(\frac{\pi}{12}\right) > 0$ and $\sin\left(\frac{\pi}{12}\right) > 0$, so

$$\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{2+\sqrt{3}}}{2} + i\frac{\sqrt{2-\sqrt{3}}}{2} \text{ then} \begin{cases} \cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{2+\sqrt{3}}}{2} \\ \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{2-\sqrt{3}}}{2} \end{cases}$$

• let $Z^3 = -8i \Rightarrow |-8i| = 8 = 2^3$ and $\arg(-8i) = \frac{3\pi}{2} + 2k\pi, k \in \mathbb{Z}$ then:

$$Z^3 = -8i \Rightarrow \begin{cases} |Z^3| = 2^3 \\ \arg(Z^3) = \frac{3\pi}{2} + 2k\pi, k \in \mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} |Z| = 2 \\ 3\arg(Z) = \frac{3\pi}{2} + 2k\pi, k \in \mathbb{Z} \end{cases}$$

$$\Leftrightarrow \begin{cases} |Z| = 2 \\ \arg(Z) = \frac{\pi}{2} + \frac{2k\pi}{3}, k \in \{0, 1, 2\} \end{cases}$$

so the cubic roots of Z are :

$k=0 \Rightarrow Z_0 = 2e^{i\frac{\pi}{2}} = 2i$

$k=1 \Rightarrow Z_1 = 2e^{i\left(\frac{\pi}{2}+\frac{2\pi}{3}\right)} = 2e^{i\left(\frac{7\pi}{6}\right)} = 2\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = -\sqrt{3} - i$

$k=2 \Rightarrow Z_2 = 2e^{i\left(\frac{\pi}{2}+\frac{4\pi}{3}\right)} = 2e^{i\left(\frac{11\pi}{6}\right)} = \sqrt{3} - i$

Exercise 4:

1) Consider the algebraic equation :$Z^2 + Z + 1 = 0$......E_1

we calculate the discriminant $\Delta = 1 - 4 = -3 = 3i^2$

the solutions of equuation are $\begin{array}{l} Z_1 = \dfrac{-1+i\sqrt{3}}{2} \\ Z_1 = \dfrac{-1-i\sqrt{3}}{2} \end{array}$

2•Let the equation $Z^2 + Z + \frac{1}{4} - \frac{3}{4}i = 0$

we calculate the discriminant $\Delta = 1 - 4\left(\frac{1}{4} - \frac{3}{4}i\right) = 3i$

$\Delta \in \mathbb{C}$,we serch the square roots of Δ

let $\omega^2 = \Delta$ with $\omega = \alpha + i\beta \Leftrightarrow (\alpha + i\beta)^2 = 3i$

so $\begin{cases} \alpha^2 - \beta^2 = 0 .... (1) \\ 2\alpha\beta = 3 .... (2) \qquad xy > 0 \Leftrightarrow \ x, y \text{ have the same sign} \\ \alpha^2 + \beta^2 = \sqrt{(0)^2 + (3)^2} = 3 ..... (3) \end{cases}$,

which gives $\alpha = \beta = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$ from where

$\omega_1 = \frac{\sqrt{6}}{2}(1+i)$ et $\omega_1 = -\frac{\sqrt{6}}{2}(1+i)$

Subseqently the sotulions of equation E_1 are

$$\begin{cases} Z_1 = \dfrac{-1 + \frac{\sqrt{6}}{2}(1+i)}{2} = \dfrac{-2+\sqrt{6}}{4} + i\dfrac{\sqrt{6}}{4} \\ Z_1 = \dfrac{-1 - \frac{\sqrt{6}}{2}(1+i)}{2} = \dfrac{-2-\sqrt{6}}{4} - i\dfrac{\sqrt{6}}{4} \end{cases}$$

3) $(1+2i)Z^2 - (9+3i)Z + 10 - 5i = 0$............$E_2$

$\Delta = (9+3i)^2 - 4(1+2i)(10-5i) = -8 - 6i$

we put $\omega^2 = \Delta$ such that $\omega = \alpha + i\beta \Leftrightarrow (\alpha + i\beta)^2 = -8 - 6i$,

by identification, we obtain the following algebraic system

$\begin{cases} \alpha^2 - \beta^2 = -8 .... (1) \\ 2\alpha\beta = -6 .... (2) \qquad xy < 0 \Leftrightarrow \ x, y \text{ different sign} \\ \alpha^2 + \beta^2 = \sqrt{(8)^2 + (6)^2} = 10 ..... (3) \end{cases}$, we find that

$\omega_1 = 1 - 3i$ and $\omega_2 = -1 + 3i$

Then the solutions of the equation E_2 are $\begin{cases} Z_1 = \dfrac{-b+\omega_1}{2a} = 2 - i \\ Z_2 = \dfrac{-b-\omega_1}{2a} = 1 - 2i \end{cases}$

4) $Z^4 - 30Z^2 + 289 = 0$..........E_3

we put $\widetilde{Z} = Z^2$ the equation E_3 becomes $\widetilde{Z}^2 - 30\widetilde{Z} + 289 = 0$.....$E_3'$

the discriminant $\Delta = (-30)^2 - 4(289) = -256 = (16i)^2$,

then the solutions of the equation E_2 are $\begin{cases} \widetilde{Z}_1 = \dfrac{30+16i}{2} = 15 + 8i \\ \widetilde{Z}_2 = \dfrac{30-16i}{2a} = 15 - 8i \end{cases}$

we search Z_1 shch that $\widetilde{Z_1} = Z_1^2 \Rightarrow Z_1^2 = 15 + 8i = 16 - 1 + 8i = (4+i)^2$

which gives $\begin{cases} Z_{1,1} = 4 + i \\ Z_{1,2} = -4 - i \end{cases}$

the same for $\widetilde{Z_2} = Z_2^2 \Rightarrow Z_1^2 = 15 - 8i = 16 - 1 - 8i = (4-i)^2$ which gives

$\begin{cases} Z_{2,1} = 4 - i \\ Z_{2,2} = -4 + i \end{cases}$

Therefore the solutions of equation E_3 are $\{4+i, -4-i, 4-i, -4+i\}$.

Exercise 5

Let the algebraic equation $Z^3 - iZ + 1 - i = 0$

1• Posons $Z = \alpha \in \mathbb{R} \Leftrightarrow \alpha^3 + 1 - i(\alpha+1) = 0 \Leftrightarrow \begin{cases} \alpha^3 + 1 = 0 \\ \alpha + 1 = 0 \end{cases}$, which give $\alpha = -1$

2• We divide $Z^3 - iZ + 1 - i$ by $(z+1)$, then $Z^3 - iZ + 1 - i = (Z+1)\left(aZ^2 + bZ + c\right)$; $a, b, c \in \mathbb{C}$

which implies $Z^3 - iZ + 1 - i = (Z+1)\left(aZ^2 + bZ + c\right) = aZ^3 + (a+b)Z^2 + (b+c)Z + c$

by identification we obtain

$\begin{cases} a = 1 \\ a + b = 0 \\ b + c = -i \\ c = 1 - i \end{cases} \Leftrightarrow \begin{cases} a = 1 \\ b = -1 \\ c = 1 - i \end{cases}$, so $Z^3 - iZ + 1 - i = (Z+1)\left(Z^2 - Z + 1 - i\right)$

so $Z^3 - iZ + 1 - i = (Z+1)\left(Z^2 - Z + 1 - i\right) = 0 \Rightarrow Z^2 - Z + 1 - i = 0$

we calculate the discriminant $\Delta = 1 - 4(1-i) = -3 + 4i = 1 + 4i - 4 = (1+2i)^2$

the roots of this polynomial are: $\begin{cases} \dfrac{1+(1+2i)}{2} = 1 + i \\ \dfrac{1-(1+2i)}{2} = -i \end{cases}$

Then the solutions of the equation E are $-1,\ 1+i,\ -i$

Exercise 6

The trigonometric form of $(1+i)^n$, $\forall n \in \mathbb{N}$

1• We have: $1+i = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{2}\exp\left(i\frac{\pi}{4}\right) = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

Then $(1+i)^n = \sqrt{2}^n \exp\left(i\frac{n\pi}{4}\right) = \sqrt{2}^n\left(\cos\left(\frac{n\pi}{4}\right) + i\sin\left(\frac{n\pi}{4}\right)\right)$ according to the formula of moivre

Therefore $(1+i)^n = (2)^{\frac{n}{2}} \exp\left(i\frac{n\pi}{4}\right)$

2• $(1-i)^n = \left(\overline{1+i}\right)^n = (2)^{\frac{n}{2}} \exp\left(-i\frac{n\pi}{4}\right)$ so

$(1+i)^n + (1-i)^n = (2)^{\frac{n}{2}}\left(\exp\left(i\frac{n\pi}{4}\right) + \exp\left(-i\frac{n\pi}{4}\right)\right) = (2)^{\frac{n}{2}}\left(2\cos\left(\frac{n\pi}{4}\right)\right)$

so

$$(1+i)^n + (1-i)^n = (2)^{\frac{n+2}{2}}\left(\cos\left(\frac{n\pi}{4}\right)\right)$$

Exercise7

Using the moivre's formula :

$\cos(4x) + i\sin(4x) = \exp(i4x) = (\cos x + i\sin x)^4$,

We expand the second membre, using the binomial formula :

so

$$(\cos x + i\sin x)^4 = (\cos x)^4 + 4(\cos x)^3(i\sin x) + 6(\cos x)^2(i\sin x)^2 + 4(\cos x)(i\sin x)^3 + (i\sin x)^4$$

$$= (\cos x)^4 - 6(\cos x)^2(\sin x)^2 + (\sin x)^4 + i4\left((\cos x)^3(\sin x) - (\cos x)(\sin x)^3\right)$$

Finaly, by identifying the real and imaginary parts, we find

$$\begin{aligned}\cos(4x) &= (\cos x)^4 - 6(\cos x)^2(\sin x)^2 + (\sin x)^4 \\ \sin(4x) &= 4(\cos x)^3(\sin x) - 4(\cos x)(\sin x)^3\end{aligned}$$