Solutions : **Serie** 3

<u>Exercise 2</u>:

By definition :

$\lim\limits_{n\to+\infty} u_n = l \Leftrightarrow \forall\varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N} : (n \geq N \Longrightarrow |u_n - l| < \varepsilon).$

1• For $\lim\limits_{n\to+\infty} \dfrac{3\sqrt{n}}{4\sqrt{n}+5} = \dfrac{3}{4}$

Let $\varepsilon > 0$, we want to prove that there exists $\exists N_\varepsilon \in \mathbb{N}$ such that $\left|\dfrac{3\sqrt{n}}{4\sqrt{n}+5} - \dfrac{3}{4}\right| < \varepsilon$; for $n \geq N_\varepsilon$.

We have : $\left|\dfrac{3\sqrt{n}}{4\sqrt{n}+5} - \dfrac{3}{4}\right| = \left|\dfrac{12\sqrt{n} - 3(4\sqrt{n}+5)}{4(4\sqrt{n}+5)}\right| = \left|\dfrac{15}{16\sqrt{n}+20}\right| < \varepsilon$, therefore $16\sqrt{n} + 20 > \dfrac{15}{\varepsilon} \Rightarrow \sqrt{n} > \dfrac{15}{16\varepsilon} - \dfrac{20}{16} \Rightarrow n > \left(\dfrac{15}{16\varepsilon} - \dfrac{5}{4}\right)^2.$

Just take $n = E\left[\left(\dfrac{15}{16\varepsilon} - \dfrac{5}{4}\right)^2\right] + 1$

2• For $\lim\limits_{n\to+\infty} \dfrac{n^2}{4n^2-1} = \dfrac{1}{4}$

$\forall\varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall n \in \mathbb{N} : (n \geq N_\varepsilon \Longrightarrow \left|\dfrac{n^2}{4n^2-1} - \dfrac{1}{4}\right| < \varepsilon.$

We have $\left|\dfrac{n^2}{4n^2-1} - \dfrac{1}{4}\right| = \left|\dfrac{4n^2 - 4n^2 + 1}{4(4n^2-1)}\right| = \left|\dfrac{1}{4(4n^2-1)}\right| < \varepsilon$, on obtient $\left(4n^2 - 1\right) > \dfrac{1}{4\varepsilon} \Rightarrow n^2 > \dfrac{1}{16\varepsilon} + \dfrac{1}{4}$ then $n > \sqrt{\dfrac{1}{16\varepsilon} + \dfrac{1}{4}}$

We take $n = E\left[\sqrt{\dfrac{1}{16\varepsilon} + \dfrac{1}{4}}\right] + 1.$

3• $\lim\limits_{n\to+\infty} \dfrac{(-1)^n}{2n+1} = 0$

Let $\varepsilon > 0$ look for $\exists N_\varepsilon \in \mathbb{N}$ such that $\left|\dfrac{(-1)^n}{2n+1}\right| < \varepsilon$; for $n \geq N_\varepsilon$.

$\forall\varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall n \in \mathbb{N} : (n \geq N_\varepsilon \Longrightarrow \left|\dfrac{(-1)^n}{2n+1}\right| < \varepsilon.$

We have $\left|\dfrac{(-1)^n}{2n+1}\right| = \dfrac{1}{2n+1} < \varepsilon$ which gives $2n+1 > \dfrac{1}{\varepsilon}$ then $n > \dfrac{1}{2}\left(\dfrac{1}{\varepsilon} - 1\right)$,just take $n = E\left[\dfrac{1}{2}\left(\dfrac{1}{\varepsilon} - 1\right)\right] + 1.$

4• $\lim\limits_{n\to+\infty} \ln(n) = +\infty.$

$\forall\ A > 0\ ,\exists N \in \mathbb{N}, \forall n \geq N \Rightarrow \ln(n) > A$

We have $\ln(n) > A \Leftrightarrow n > e^A$

we take $N = E\left(e^A\right) + 1$

.**<u>Exercise 1</u> : study of convergence**

a sequence $(U_n)_{n\in IN}$ converges if and only if $\lim\limits_{n\to+\infty} U_n = l$ finite

We find the limit of a sequence when $n \to +\infty$

1) $B_n = \dfrac{\sqrt{n} - n + 1}{2\sqrt{n} + n + 2}$

$\lim\limits_{n \to +\infty} B_n = \lim\limits_{n \to +\infty} \dfrac{\sqrt{n} - n + 1}{2\sqrt{n} + n + 2} = \lim\limits_{n \to +\infty} \dfrac{-n}{n} = -1 \Rightarrow (B_n)_{n \in IN}$ converges to -1

2) $D_n = \dfrac{1! + 2! + ... + (n+1)!}{(n+1)!}$, we note that $1! + 2! + ... + n! + (n+1)! \leq (n-1)(n-1)! + n! + (n+1)!$

then $1 \leq D_n \leq \dfrac{(n-1)(n-1)! + n! + (n+1)!}{(n+1)!} = \dfrac{(n-1)(n-1)!}{(n+1)!} + \dfrac{n!}{(n+1)!} + 1$

and $1 \leq \lim\limits_{n \to +\infty} D_n \leq \lim\limits_{n \to +\infty} \dfrac{(n-1)(n-1)!}{(n+1)!} + \dfrac{n!}{(n+1)!} + 1 = 1$ therfore $\lim\limits_{n \to +\infty} D_n = 1$ and the sequence $(u_n)_n$ converges to 1

3) Let $C_n = t_n \dfrac{1}{2 + \sin\sqrt{n}}$ such that $t_n = \sqrt{n+1} - \sqrt{n}$

we have $t_n = \dfrac{\left(\sqrt{n+1} - \sqrt{n}\right)\left(\sqrt{n+1} + \sqrt{n}\right)}{\sqrt{n+1} + \sqrt{n}} = \dfrac{1}{\sqrt{n+1} + \sqrt{n}}$ and $\lim\limits_{n \to +\infty} T_n = 0$

on the other hand $(-1 \leq \sin\sqrt{n} \leq 1)$ which gives $\dfrac{1}{3} \leq \dfrac{1}{2 + \sin\sqrt{n}} \leq 1$ (bounded) *then* $\lim\limits_{n \to +\infty} C_n = 0$

hence the sequence $(C_n)_n$ converges to 0

Exercise 3

Study of the monotonicity of (u_n)

$$u_{n+1} - u_n = \left(\frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} + ... + \frac{1}{2n+1} + \frac{1}{2n+2}\right) - \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + ... +\right.$$

$$= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{(2n+2) + (2n+1) - 2(2n+1)}{(2n+1)(2n+2)} = \frac{1}{(2n+1)(2n+2)} > 0 \Rightarrow$$

$u_{n+1} - u_n > 0 \Leftrightarrow u_{n+1} > u_n$.

So the sequence (u_n) is increasing.

2- We have : $\forall k \in \{1, 2.3, ..., n\}$

$$\frac{1}{2n} \leq \frac{1}{n+k} < \frac{1}{n+1}$$

We obtain

$$\underbrace{\frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n} .... + \frac{1}{2n}}_{n \text{ fois}} \leq \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + ... + \frac{1}{2n} < \underbrace{\frac{1}{n+1} + \frac{1}{n+1} + \frac{1}{n+1} + ... + \frac{1}{n+1}}_{n \text{ fois}}$$

Which give

$$\frac{n}{2n} \le u_n < \frac{n}{n+1}$$

this leads to

$$\frac{1}{2} \le u_n < 1$$

the sequence (u_n) is bounded above by 1 and increasing so it converges to a finite limit l such that $\frac{1}{2} \le l < 1$

Exercise 4:- •Let $x_n = \frac{1 \times 3 \times 5 \times ...(2n-1)}{2 \times 4 \times 6 \times ...2n}$,

it is clear that $x_n > 0, \forall n \in \mathbb{N}^*$, then the sequence (x_n) is bounded above by 0,

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for the study of the monotony (x_n), just compare the quotient $\frac{x_{n+1}}{x_n}$ by 1

$\frac{x_{n+1}}{x_n} = \frac{1 \times 3 \times 5 \times ...(2n+1)}{2 \times 4 \times 6 \times ...2n \times (2n+2)} \times \frac{2 \times 4 \times 6 \times ...2n}{1 \times 3 \times 5 \times ...(2n-1)} = \frac{(2n+1)}{(2n+2)} < 1.$

so the sequence (x_n) is decreasing and bounded below so it converges

•Let $y_n = \sum_{k=1}^{n} \frac{1}{k^p}$ $(p \ge 2)$

Study of monotony

we have $u_{n+1} = \sum_{k=1}^{n+1} \frac{1}{k^p} = u_n + \frac{1}{(n+1)^p} > u_n$, so (u_n) is strictly increasing.

On the other hand, we have

$$\begin{aligned} u_n = 1 + \frac{1}{2^P} + \frac{1}{3^P} + ... + \frac{1}{n^p} < 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} \\ < 1 + \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + ... + \frac{1}{(n-1) \times n} \\ = 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + ... \frac{1}{n-1} \\ < 2 - \frac{1}{n} < 2 \end{aligned}$$

therefore (u_n) is bounded above;

As (u_n) is an increasing and bounded above ,it is therefore convergent

Exercise 5

let (u_n) defined by $\begin{cases} u_0 = 1 \\ u_{n+1} = \dfrac{3u_n + 2}{u_n + 2} \end{cases}$

1-We write (u_{n+1}) in the form $u_{n+1} = a + \frac{b}{u_n + 2}$. $\Rightarrow u_{n+1} = \frac{au_n + 2a + b}{u_n + 2}$

by identification we find that $\begin{cases} a = 3 \\ 2a + b = 2 \Rightarrow b = -4 \end{cases}$

hence $u_{n+1} = 3 - \frac{4}{u_n + 2}$

2- Let us show by induction that : $\forall n \in \mathbb{N}, : 0 < u_n < 2$.

For $n = 0$ $0 < u_0 < 2$

We assume that it is true for n and we show that it is true for $n+1$
we have
$0 < u_n < 2. \Rightarrow 2 < u_n + 2 < 4 \Rightarrow -2 < -\dfrac{4}{u_n+2} < -1 \Rightarrow 0 < 1 < 3 - \dfrac{4}{u_n+2} < 2$

hence $0 < u_{n+1} < 2.$

3-Study of monotony $\Rightarrow$
we use the recurrent sequence with
$f(x) = 3 - \dfrac{4}{x+2} \Rightarrow f' > 0 \Rightarrow (u_n)_{n\in\mathbb{N}}$ is monotonic and $sig(u_{n+1} - u_n) = sig(u_1 - u_0) > 0$

$$u_{n+1} - u_n = \frac{3u_n+2}{u_n+2} - u_n = \frac{-u_n^2 + u_n + 2}{u_n+2} = \frac{(2-u_n)(1+u_n)}{u_n+2} > 0 \text{ in }]0,2[\qquad \text{then } (u_n)_{n\in\mathbb{N}} \text{ es}$$

4-Let us show that for all $n \in \mathbb{N}$, : $|u_{n+1} - 2| < \frac{1}{2}|u_n - 2|$.

we have $|u_{n+1} - 2| = \left|3 - \dfrac{4}{u_n+2} - 2\right| = \left|1 - \dfrac{4}{u_n+2}\right| = \left|\dfrac{u_n-2}{u_n+2}\right|$ but $\dfrac{1}{u_n+2} < \dfrac{1}{2}$,

we therefore obtain $|u_{n+1} - 2| = \left|\dfrac{u_n-2}{u_n+2}\right| < \dfrac{1}{2}|u_n - 2|$,

5) Deduce that $\forall n \in \mathbb{N}$, we have : $|u_n - 2| < \left(\dfrac{1}{2}\right)^n |u_0 - 2|$.

according to question 4 we have $|u_{n+1} - 2| < \dfrac{1}{2}|u_n - 2| \Rightarrow$

$|u_n - 2| < \dfrac{1}{2}|u_{n-1} - 2| < \left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right)|u_{n-2} - 2| \ldots. < \left(\dfrac{1}{2}\right)^n |u_0 - 2|$ (we can prove by induction)

6) From question 5, we can deduce that the sequence $(u_n)_{n\in\mathbb{N}}$ converges to $l = 2$

Exercise7

we show that (u_n) and (v_n) are two adjacent sequences

such that $\begin{cases} u_{n+1} = \dfrac{2u_nv_n}{u_n+v_n} \\ v_{n+1} = \dfrac{u_n+v_n}{2} \end{cases}$

• We first show that $\forall n \in \mathbb{N}$,$u_n < v_n$
By induction
for $n = 0$ $\quad u_0 < v_0$ is true
We assume that it is true for (n) and show that it is true for $(n+1)$
We have $\quad u_n < v_n$

then $v_{n+1} - u_{n+1} = \dfrac{u_n+v_n}{2} - \dfrac{2u_nv_n}{u_n+v_n}$

$= \dfrac{(u_n+v_n)^2 - 4u_nv_n}{2(u_n+v_n)} = \dfrac{(u_n-v_n)^2}{2(u_n+v_n)} > 0$ hence $v_{n+1} > u_{n+1}$

•study of monotony

a) $u_{n+1}-u_n=\dfrac{2u_nv_n}{u_n+v_n}-u_n=\dfrac{2u_nv_n-u_n^2-u_nv_n}{u_n+v_n}$

$=\dfrac{u_nv_n-u_n^2}{u_n+v_n}=\dfrac{u_n(v_n-u_n)}{u_n+v_n}>0\Leftrightarrow (u_n)$ is strictly increasing

b) $v_{n+1}-v_n=\dfrac{u_n+v_n}{2}-v_n=\dfrac{u_n-v_n}{2}<0\quad (u_n<v_n)\Leftrightarrow (v_n)$ is strictly decreasing

As (u_n) is strictly increasing and bounded above by v_0 thereforeonc (u_n) is convergent, it admits a finite limit . The same for the sequence (v_n) is strictly decreasing and bounded below by u_0 then (v_n) is convergent , it has a finite limit

- We set that $\lim\limits_{n\to+\infty}u_n=l$ and $\lim\limits_{n\to+\infty}v_n=l'$ then

$l=\dfrac{2ll'}{l+l'}$ and $l'=\dfrac{l+l'}{2}\Rightarrow 2l'=l+l'\Leftrightarrow l=l'$ and $\lim\limits_{n\to+\infty}(u_n-v_n)=l-l'=0$

So (u_n) and (v_n) are adjacent

Exercise 8

(u_n) is not a cauchy sequence $\Leftrightarrow \overline{\forall\varepsilon>0,\exists N\in\mathbb{N}^* \text{ such that } \forall p,q\in\mathbb{N},p,q\geq: N\Rightarrow |u_p-u_q|<\varepsilon}$

$\Leftrightarrow \exists\varepsilon>0,\forall N\in\mathbb{N}^*$ such that $\exists p,q\in\mathbb{N}:p,q\geq: N\wedge|u_p-u_q|\geq\varepsilon.$

Let us show by induction that $\forall N\geq 1$ we have $u_{2N}-u_N\geq\dfrac{1}{2}$.

For $N=1$we have $u_{2N}-u_N=u_2-u_1=\left(1+\dfrac{1}{2}\right)-1=\dfrac{1}{2}\geq\dfrac{1}{2}$

Assume that it is true for N and we show that it is true for $N+1$

we have

$u_{2(N+1)}-u_{N+1}=\left(1+\dfrac{1}{2}+...\dfrac{1}{N+1}+....\dfrac{1}{2N}+\dfrac{1}{2N+1}+\dfrac{1}{2(N+1)}\right)-\left(1+\dfrac{1}{2}+..+\dfrac{1}{N}+\dfrac{1}{N+1}\right)$

$=(u_{2N}-u_N)+\dfrac{1}{2N+1}+\dfrac{1}{2(N+1)}-\dfrac{1}{N+1}\geq u_{2N}-u_N$

$\Rightarrow u_{2(N+1)}-u_{N+1}\geq\dfrac{1}{2}$

hence $\forall N\in\mathbb{N}^*: u_{2N}-u_N\geq\dfrac{1}{2}$.

Which implies that the sequence (u_n) is not a cauuchy sequence.Indeed, we set $\varepsilon=\dfrac{1}{2}>0$

Let $N\in\mathbb{N}^*$, we put $p=2N$, $q=N$, such that $p=2N\geq N$

and $q=N\geq N$.We obtain:$|u_p-u_q|=|u_{2N}-u_N|=u_{2N}-u_N\geq\dfrac{1}{2}=\varepsilon.$

Then $\exists\varepsilon>0,\forall N\in\mathbb{N}^*$ such that $\exists p=2N,q=N\in\mathbb{N}^*:p,q\geq: N\wedge|u_p-u_q|\geq\varepsilon=\dfrac{1}{2}.$