Solutions

Exercise 1: using the definition of the limit

1) $\lim\limits_{x\to 1} 3x+3=6.$ iff

$\forall\varepsilon>0,\quad \exists\delta>0,\quad \forall x\in D_f : |x-x_0|<\delta \Longrightarrow |f(x)-l|<\varepsilon$

so, $\forall\varepsilon>0,\quad \exists\delta>0,\quad \forall x\in \mathbb{R} : |x-1|<\delta \Longrightarrow |3x+3-6|<\varepsilon$

then $|3x-3|<\varepsilon \Longleftrightarrow 3|x-1|<\varepsilon \Longleftrightarrow |x-1|<\frac{\varepsilon}{3}.$

So, just take:$\delta=\dfrac{\varepsilon}{3}.$

2) $\lim\limits_{x\to 0}\dfrac{2x-3}{3x+1}=-3.$ iff $\forall\varepsilon>0,\ \exists\delta>0,\ \forall x\in D_f : |x-x_0|<\delta \Longrightarrow |f(x)-l|<\varepsilon$

$\forall\varepsilon>0,\ \exists\delta>0,\ \forall x\in \mathbb{R}\setminus\left\{\frac{1}{3}\right\} : |x|<\delta \Longrightarrow \left|\frac{2x-3}{3x-1}+3\right|<\varepsilon$

we obtain $\left|\dfrac{2x-3+9x+3}{3x+1}\right|<\varepsilon \Longrightarrow \left|\dfrac{11x}{3x+1}\right|<\varepsilon$

We have the neighborhood of 0 (υ_0) , $\left]0-\frac{1}{4},0+\frac{1}{4}\right[=\left]-\frac{1}{4},\frac{1}{4}\right[.$

therefore

$-\frac{1}{4}<x<\frac{1}{4}\Longrightarrow -\frac{3}{4}<3x<\frac{3}{4}\Longrightarrow \dfrac{1}{4}<3x+1<\dfrac{7}{4}\Rightarrow \dfrac{4}{7}<\dfrac{1}{3x+1}<4$

then $\left|\frac{1}{3x-1}\right|<4...(1)$,

on the other hand, we have $|11x|<\varepsilon \Longrightarrow |x|<\dfrac{\varepsilon}{11}...(2)$

according to (1) and (2) : $\left|\dfrac{11x}{3x+1}\right|=\left|\frac{1}{3x+1}\right|.|11x|<4.\dfrac{\varepsilon}{11}=\dfrac{4\varepsilon}{11}$ so just take $\delta=\frac{4\varepsilon}{11}$

3) $\lim\limits_{x\to 0}\frac{1}{x^2}=+\infty.$ iff $\forall A>0,\ \exists\delta>0,\ \forall x\in D_f : |x-x_0|<\delta \Longrightarrow f(x)>A$

so $\forall A>0,\ \exists\delta>0,\ \forall x\in \mathbb{R}^* : |x|<\delta \Longrightarrow \frac{1}{x^2}>A$

we have $\frac{1}{x^2}>A\Longrightarrow x^2<\frac{1}{A}\Longrightarrow x<\sqrt{\frac{1}{A}}$ then , just take $\delta=\sqrt{\dfrac{1}{A}}.$

4) $\lim\limits_{x\to +\infty} x^2+x+1=+\infty.$ iff $\forall A>0,\ \exists B>0,\ \forall x\in D_f : x>B \Longrightarrow f(x)>A$

So $\forall A>0,\ \exists B>0,\ \forall x\in \mathbb{R} : x>B \Longrightarrow x^2+x+1>A$ we have $x^2+x+1>x>A.$

so , just take $B=A.$

$b)$ We show that

$\lim\limits_{x\to 0}\cos\left(\frac{1}{x}\right)$ does not exist, it is enough to consider the two sequences:

$u_n=\dfrac{1}{2n\pi}$ and $v_n=\frac{1}{(2n+1)\pi}$

we obtain

$$\lim_{n\to+\infty} u_n=\lim_{n\to+\infty}\frac{1}{2n\pi}=0 \qquad \text{and} \qquad \lim_{x\to 0} v_n=\lim_{n\to+\infty}\frac{1}{(2n+1)\pi}=0$$

but $\lim\limits_{n\to+\infty}\cos\frac{1}{u_n}=1$ et $\lim\limits_{n\to+\infty}\cos\frac{1}{v_n}=-1.$

So $\lim\limits_{x\to 0}\cos\left(\frac{1}{x}\right)$ does not exist

Exercise 2: Calculate the limits

1) • $\lim\limits_{x\to a}\dfrac{x^2-a^2}{x^3-a^3}=\dfrac{0}{0}$ I.F (indeterminate form)

We have $\left(x^2-a^2\right)=(x-a)(x+a)$ and $\left(x^3-a^3\right)=(x-a)\left(x^2+ax+a^2\right)$ then

$$\lim_{x\to a}\frac{x^2-a^2}{x^3-a^3}=\lim_{x\to a}\frac{(x-a)(x+a)}{(x-a)\left(x^2+ax+a^2\right)}=\lim_{x\to a}\frac{(x+a)}{\left(x^2+ax+a^2\right)}=\frac{2a}{3a^2}=\frac{2}{3a}$$

Hence $\lim\limits_{x\to a}\dfrac{x^2-a^2}{x^3-a^3}=\dfrac{2}{3a}$

2)• $\lim\limits_{x\to 0}\dfrac{\cos(ax)-\cos(bx)}{x^2}=\dfrac{0}{0}$ I.F

We have $\cos ax-\cos bx=-2\sin\left(\dfrac{(a+b)x}{2}\right)\sin\left(\dfrac{(a-b)x}{2}\right)$

Then

$$\lim_{x\to 0}\frac{\cos(ax)-\cos(bx)}{x^2}=\lim_{x\to 0}\frac{-2\sin\left(\frac{(a+b)x}{2}\right)\sin\left(\frac{(a-b)x}{2}\right)}{x^2}=\lim_{x\to 0}-2\frac{\sin\left(\frac{(a+b)x}{2}\right)}{x}\times\frac{\sin\left(\frac{(a-b)x}{2}\right)}{x}$$

$$=-2\lim_{x\to 0}\frac{(a+b)}{2}\frac{\sin\left(\frac{(a+b)x}{2}\right)}{\frac{(a+b)x}{2}}\times\lim_{x\to 0}\frac{(a-b)}{2}\frac{\sin\left(\frac{(a-b)x}{2}\right)}{\frac{(a-b)x}{2}}$$

we know that $\lim\limits_{u\to 0}\dfrac{\sin u}{u}=1$, then we obtain $\lim\limits_{x\to 0}\dfrac{\cos(ax)-\cos(bx)}{x^2}=-2\dfrac{(a+b)}{2}\times\dfrac{(a-b)}{2}=\dfrac{b^2-a^2}{2}$

hence $\lim\limits_{x\to 0}\dfrac{\cos(ax)-\cos(bx)}{x^2}=\dfrac{b^2-a^2}{2}$

3) • $\lim\limits_{x\to+\infty}x^2\left(1-\cos\frac{1}{x}\right)=+\infty-\infty$ I.F

We set $X=\dfrac{1}{x}$ when $x\to+\infty\Rightarrow X\to 0$ hence $\lim\limits_{x\to+\infty}x^2\left(1-\cos\frac{1}{x}\right)$ becomes $\lim\limits_{X\to 0}\frac{1}{X^2}(1-\cos X)$

we use the equivalence functions in the neighborhood of $X_0=0$ $\cos X\underset{0}{\sim}1-\dfrac{X^2}{2}$, then

$$\lim_{x\to+\infty}x^2\left(1-\cos\tfrac{1}{x}\right)=\lim_{X\to 0}\tfrac{1}{X^2}(1-\cos X)=\lim_{X\to 0}\tfrac{1}{X^2}\left(\frac{X^2}{2}\right)=\frac{1}{2},$$

hence $\lim\limits_{x\to+\infty}x^2\left(1-\cos\frac{1}{x}\right)=\dfrac{1}{2}$

4) • $\lim\limits_{x\to+\infty}E\left(\dfrac{\ln(x)}{x}\right)=?$

by definition $x-1<E(x)\le x$, then $\dfrac{\ln(x)}{x}-1<E\left(\dfrac{\ln(x)}{x}\right)\le\dfrac{\ln(x)}{x}$

hence $\lim\limits_{x\to+\infty}\left(\dfrac{\ln(x)}{x}-1\right)<\lim\limits_{x\to+\infty}E\left(\dfrac{\ln(x)}{x}\right)\le\lim\limits_{x\to+\infty}\dfrac{\ln(x)}{x}$

we know that $\lim\limits_{x\to+\infty}\dfrac{\ln(x)}{x}=0$, then we obtain $-1<\lim\limits_{x\to+\infty}E\left(\dfrac{\ln(x)}{x}\right)\le 0$ hence

$\lim\limits_{x\to+\infty} E\left(\dfrac{\ln(x)}{x}\right) = 0$.

5) $\bullet \lim\limits_{x\to 3}\dfrac{|x-3|}{x-3} = \dfrac{0}{0}$ I.F

We have $|x-3| = \begin{cases} -x+3 & \text{si } x<3 \\ x-3 & \text{si } x\geq 3 \end{cases}$

Which give $\lim\limits_{x\overset{<}{\to}3}\dfrac{|x-3|}{x-3} = \lim\limits_{x\overset{<}{\to}3}\dfrac{-x+3}{x-3} = \lim\limits_{x\overset{<}{\to}3}\dfrac{-(x-3)}{x-3} = -1$

and

$\lim\limits_{x\overset{>}{\to}3}\dfrac{|x-3|}{x-3} = \lim\limits_{x\overset{>}{\to}3}\dfrac{x-3}{x-3} = 1$

Consequensly $\lim\limits_{x\overset{<}{\to}3}\dfrac{|x-3|}{x-3} \neq \lim\limits_{x\overset{>}{\to}3}\dfrac{|x-3|}{x-3}$ then the function $\dfrac{|x-3|}{x-3}$ does not admit a limit at the point $x_0 = 3$.

6) $\bullet \lim\limits_{x\to+\infty}\left(\dfrac{x-1}{x+1}\right)^x = 1^{+\infty}$ FI

Let us remember that $\lim\limits_{X\to+\infty}\left(1+\frac{1}{X}\right)^X = \lim\limits_{X\to 0}(1+X)^{\frac{1}{X}} = e$

then , we writte $\left(\dfrac{x-1}{x+1}\right)^x$ at the form $\left(1+\frac{1}{X}\right)^X$.

Let $\dfrac{x-1}{x+1} = 1-\dfrac{2}{x+1} = 1+\dfrac{1}{\dfrac{-(x+1)}{2}}$, then $\lim\limits_{x\to+\infty}\left(\dfrac{x-1}{x+1}\right)^x = \lim\limits_{x\to+\infty}\left(1+\dfrac{1}{\dfrac{-(x+1)}{2}}\right)^x$

we use a change of variable or we put $X = \dfrac{-(x+1)}{2}$ then $x = -2X-1$.

we obtaint

$$\lim_{x\to+\infty}\left(\frac{x-1}{x+1}\right)^x = \lim_{X\to-\infty}\left(1+\frac{1}{X}\right)^{-2X-1.} = \lim_{X\to-\infty}\left[\left(1+\frac{1}{X}\right)^X\right]^{-2} \times \left(1+\frac{1}{X}\right)^{-1} = e^2$$

then $\lim\limits_{x\to+\infty}\left(\dfrac{x-1}{x+1}\right)^x = e^2$

7) $\lim\limits_{x\to+\infty}\dfrac{x}{2}\ln\left(\sqrt{1+\dfrac{1}{x}}\right) = +\infty\times 0$ I.F

$$\lim_{x\to+\infty}\frac{x}{2}\ln\left(\sqrt{1+\frac{1}{x}}\right) = \lim_{x\to+\infty}\frac{x}{2}\ln\left(1+\frac{1}{x}\right)^{\frac{1}{2}} = \lim_{x\to+\infty}\frac{x}{4}\ln\left(1+\frac{1}{x}\right)$$

we put $X = \dfrac{1}{x}$ as $x\to+\infty \Rightarrow X\to 0$ we obtain

$$\lim_{X\to 0}\frac{1}{4X}\ln(1+X) = \begin{cases} \textbf{method1: } \lim\limits_{X\to 0}\dfrac{1}{4X}\ln(1+X) = \lim\limits_{X\to 0}\frac{1}{4}\ln(1+X)^{\frac{1}{X}} = \frac{1}{4}\ln e = \frac{1}{4} \\ \textbf{method 2:}\text{we use the equivalence functions } \ln(1+X)\underset{0}{\sim} X \\ \quad\text{which give } \lim\limits_{X\to 0}\dfrac{1}{4X}\ln(1+X) = \lim\limits_{X\to 0}\dfrac{1}{4X}X = \frac{1}{4} \end{cases}$$

Exercise 3:

We study the continuity of the following functions

$$\bullet f(x) = \begin{cases} \dfrac{5x^2-2}{2}, & x \geq 1 \\ \cos(x-1), & x \leq 1 \end{cases}$$

the function $f_1(x) = \dfrac{5x^2-2}{2}$ is continuous on $\mathbb{R}$so it is continuous sur $[1, +\infty[$.

et $f_2(x) = \cos(x-1)$ is continuous on $\mathbb{R}$ so it is continuous on $]-\infty, 1]$.

We study the continuity of function f at point $x_0 = 1$ that's to find $\lim\limits_{x \overset{\leq}{\to} 1} f(x) \overset{?}{=} \lim\limits_{x \overset{\geq}{\to} 1} f(x) \overset{?}{=} f(x_0)$,

we have $\lim\limits_{x \overset{\leq}{\to} 1} f(x) = \lim\limits_{x \overset{\leq}{\to} 1} \cos(x-1) = \cos 0 = 1$

and $\lim\limits_{x \overset{\geq}{\to} 1} f(x) = \lim\limits_{x \overset{\geq}{\to} 1} \dfrac{5x^2-2}{2} = \dfrac{3}{2}$

But $\lim\limits_{x \overset{\geq}{\to} 1} f(x) \neq \lim\limits_{x \overset{\leq}{\to} 1} f(x)$ then the function f does not continuous at $x_0 = 1$ (f est discontinuous) $\Rightarrow f$ is continuous on $\mathbb{R}/\{1\}$.

$$\bullet g(x) = \begin{cases} x^n \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad n \in \mathbb{N}$$

we have $x^n \sin\left(\frac{1}{x}\right)$ is continuous on $\mathbb{R}^*$and $g(0) = 0$

we seek for the continuity of $g(x)$ at point $x_0 = 0$

• If $n = 0 \Rightarrow$

$$g(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$\lim\limits_{x \to 0} g(x) = \lim\limits_{x \to 0} \sin\left(\frac{1}{x}\right)$ this limit does not exist then the function $g(x)$ does not continuous at point $x_0 = 0$

• If $n \neq 0 \Rightarrow$

$$g(x) = \begin{cases} x^n \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases},$$

and $\lim\limits_{x \to 0} g(x) = \lim\limits_{x \to 0} x^n \sin\left(\frac{1}{x}\right) = 0 \left(x^n \underset{x \to 0}{\to} 0 \text{ and } \left|\sin\left(\frac{1}{x}\right)\right| \leq 1 \right)$

we have $\lim\limits_{x \to 0} g(x) = g(0) = 0$ then the function $g(x)$ is continuous at point 0,as it is continuous on $\mathbb{R}^*$ so $g(x)$ is continuous on $\mathbb{R}$.

Exercise 4:

We find the values of α and β so that the functions f, g, and h are continuous on $\mathbb{R}$

• $f(x) = \begin{cases} x+1 & x \leq 1 \\ 3-\alpha x^2 & x \geq 1 \end{cases} \quad f(1) = 2$;for f to be continuous on $\mathbb{R}$, it must be continuous at the point $x_0 = 1$

$f(x)$ is continuous at x_0 iff

$$\lim_{x \overset{\leq}{\to} 1} f(x) = \lim_{x \overset{\geq}{\to} 1} f(x) = f(1) = 2 \Rightarrow \lim_{x \overset{\leq}{\to} 1} x + 1 = \lim_{x \overset{\geq}{\to} 1} \left(3 - \alpha x^2\right) \Rightarrow 2 = 3 - \alpha$$

we get $\boxed{\alpha = 1}$

$f(x)$ is continuous on $\mathbb{R}$ for $\alpha = 1$

• $g(x) = \begin{cases} \alpha x + \beta & x \leq 0 \\ \dfrac{1}{x+1} & x \geq 0, \qquad g(0) = 1 \end{cases}$;for g to be continuous on $\mathbb{R}$,it must be continuous at the point $x_0 = 0$.

$g(x)$ is continuous at x_0 iff

$$\lim_{x \overset{\leq}{\to} 0} g(x) = \lim_{x \overset{\geq}{\to} 0} g(x) = g(0) = 1 \Rightarrow \boxed{\beta = 1} et \forall \alpha \in \mathbb{R}.$$

$g(x)$ is continuous on $\mathbb{R}$ for $\boxed{\beta = 1}$ and $\forall \alpha \in \mathbb{R}$.

• $h(x) = \begin{cases} \dfrac{\sqrt{1+x}-1}{x} & \text{si } x \in [-1, 0[\cup]0, +\infty[\\ \alpha & \text{si } x = 0; \qquad h(0) = \alpha \end{cases}$; $h(x)$ is continuous on $[-1, 0[\cup]0, +\infty[$,

we study the continuity of $h(x)$ at point $x_0 = 0$

$h(x)$ is continuous at point x_0 iff

$$\lim_{x \to 0} h(x) = \lim_{x \to 0} h(x) = h(0) = \alpha \Rightarrow \lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} = \frac{0}{0} \ I.F$$

$$\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x \to 0} \frac{\left(\sqrt{1+x}-1\right)\left(\sqrt{1+x}+1\right)}{x\left(\sqrt{1+x}+1\right)} = \lim_{x \to 0} \frac{1+x-1}{x\left(\sqrt{1+x}+1\right)} = \lim_{x \to 0} \frac{x}{x\left(\sqrt{1+x}+1\right)}$$

$$= \lim_{x \to 0} \frac{1}{\left(\sqrt{1+x}+1\right)} = \frac{1}{2} = h(0) = \alpha \Rightarrow \boxed{\alpha = \frac{1}{2}}$$

As a result, $h(x)$ is continuous on $[1, +\infty[$ for $\alpha = \dfrac{1}{2}$

<u>Exercise 5:</u>

Study of the continuity extension of the following functions:

1) $f(x) = \dfrac{|x|}{x}$.

$f(x)$ does not defined in 0;

In addition $\lim_{x \overset{\geq}{\to} 0} f(x) = 1$ and $\lim_{x \overset{\leq}{\to} 0} f(x) = -1$

therefore $\lim_{x \to 0} f(x)$ does not exist , then $f(x)$ is not extendable by continuity in 0.

2) $g(x) = \dfrac{1 - \cos\sqrt{|x|}}{|x|}$.

$g(x)$ is not defined in 0; In addition

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{1 - \cos\sqrt{|x|}}{|x|} = \lim_{x \to 0} \frac{1 - \cos\sqrt{|x|}}{|x|} \cdot \frac{1 + \cos\sqrt{|x|}}{1 + \cos\sqrt{|x|}} = \lim_{x \to 0} \frac{1 - \cos^2\sqrt{|x|}}{|x| \cdot \left(1 + \cos\sqrt{|x|}\right)}$$

$$= \lim_{x \to 0} \frac{\sin^2\sqrt{|x|}}{\left(\sqrt{|x|}\right)^2 \cdot \left(1 + \cos\sqrt{|x|}\right)} = \lim_{x \to 0} \left(\frac{\sin\sqrt{|x|}}{\left(\sqrt{|x|}\right)\cdot}\right)^2 \cdot \frac{1}{\left(1 + \cos\sqrt{|x|}\right)} = \frac{1}{2}.$$

So $\lim_{x \to 0} g(x) = \dfrac{1}{2}$, then $g(x)$ is extendable by continuity to 0. Its extension function is:

$$G(x) = \widetilde{g}(x) \begin{cases} \dfrac{1 - \cos\sqrt{|x|}}{|x|} & \text{if} \quad x \neq 0. \\ \frac{1}{2} & \text{if} \quad x = 0. \end{cases}$$

3) $h(x) = \dfrac{(x-1)\sin x}{2x^2-2}$.

• $h(x)$ does not defined in $+1$ and in -1,

in addition

$$\lim_{x\to 1} h(x) = \lim_{x\to 1} \frac{(x-1)\sin x}{2x^2-2} = \lim_{x\to 1} \frac{(x-1)\sin x}{2(x-1)(x+1)} = \lim_{x\to 1} \frac{\sin x}{2(x+1)} = \frac{\sin(1)}{4}$$

$$= \lim_{x\to -1} h(x) = \lim_{x\to -1} \frac{(x-1)\sin x}{2x^2-2} = \pm\infty.$$

In conclusion, $h(x)$ can be extended by continuity in $x_0 = 1$, but is not extendable in $x_1 = 1$ and its extension is written:

$$H(x) = \tilde{h}(x) = \begin{cases} \dfrac{(x-1)\sin x}{2x^2-2} & \text{if } x \neq 1, \\ \dfrac{\sin(1)}{4} & \text{if } x = 1. \end{cases}$$

<u>Exercise 6:</u>

I-1) we have $f(x) = 1 + \sin x - x^2$ a definite and continuous function on $[0, \pi]$.

In addition , we have

$f(0) = 1$ and $f(\pi) = 1 - \pi^2 < 0$. then $f(0).f(\pi) < 0$

Then, according to the intermediate value theorem: the equation $f(x) = 0$ admits at least one solution in $[0, \pi]$.

2) $f(x) = x^3 - 3x - 3$ a definite and continuous function on $[2, 3]$.

In addition , we have $f(2) = -1$ and $f(3) = 15$. then $f(2).f(3) < 0$

Then, according to the intermediate value theorem: the equation $f(x) = 0$ admits at least one solution in $]2, 3[$

II- For $y \geq x \geq 0$, $\left(\sqrt{y} - \sqrt{x}\right)^2 = y + x - 2\sqrt{xy} \leq y - x$

so $\sqrt{y} - \sqrt{x} \leq \sqrt{y - x}$

By symetric $\forall x, y \geq 0$, $\left|\sqrt{y} - \sqrt{x}\right| \leq \sqrt{|y - x|}$

Let $\varepsilon > 0$. Consideration is given to $\eta = \varepsilon^2 > 0$.

for all $x, y \geq 0$, $|y - x| \leq \eta \Longrightarrow \left|\sqrt{y} - \sqrt{x}\right| \leq \sqrt{|y - x|} \leq \sqrt{\eta} = \varepsilon$.

the function of square root is therefore uniformly continuous.