Solutions

Exercise 1

the differentiability at the point $x_0 = -1$

$1\bullet f(x) = x^2 + |x+1|\,,\ \ x_0 = -1, 1$

we have $f(x) = \begin{cases} x^2 + x + 1 & \text{if } x \geq -1 \\ x^2 - x - 1 & \text{if } \ x < -1 \end{cases}$ $\quad , f(1) = 3; \quad f(-1) = 1$

$f(x)$ is differentiable (*derivable*) at $x_0 = 1$ iff $\lim\limits_{x \overset{>}{\to} 1} \dfrac{f(x) - f(1)}{x-1} = \lim\limits_{x \overset{<}{\to} 1} \dfrac{f(x) - f(1)}{x-1} =$ l finished

$\lim\limits_{x \overset{>}{\to} 1} \dfrac{f(x) - f(1)}{x-1} = \lim\limits_{x \overset{>}{\to} 1} \dfrac{x^2 + x + 1 - 3}{x-1} = \lim\limits_{x \overset{>}{\to} 1} \dfrac{x^2 + x - 2}{x-1} = \lim\limits_{x \overset{>}{\to} 1} \dfrac{(x+2)(x-1)}{(x-1)} =$

$\lim\limits_{x \overset{>}{\to} 1} (x+2) = 3 = \lim\limits_{x \overset{<}{\to} 1} \dfrac{f(x) - f(1)}{x-1} = f'(1)$

from where f is derivable at $x_0 = 1$

the derivabity at $x_0 = -1$

$f(x)$ is derivable at $x_0 = 1$ iff $\lim\limits_{x \overset{>}{\to} -1} \dfrac{f(x) - f(-1)}{x-1} = \lim\limits_{x \overset{<}{\to} -1} \dfrac{f(x) - f(-1)}{x-1} = l'$ finite

$\triangleright \lim\limits_{x \overset{>}{\to} -1} \dfrac{f(x) - f(-1)}{x+1} = \lim\limits_{x \overset{>}{\to} -1} \dfrac{x^2 + x + 1 - 1}{x+1} = \lim\limits_{x \overset{>}{\to} -1} \dfrac{x^2 + x}{x+1} = \lim\limits_{x \overset{>}{\to} -1} \dfrac{x(x+1)}{x+1} =$ $-1 = f_r'(-1)$

$\triangleright \lim\limits_{x \overset{<}{\to} -1} \dfrac{f(x) - f(-1)}{x+1} = \lim\limits_{x \overset{<}{\to} -1} \dfrac{x^2 - x - 1 - 1}{x+1} = \lim\limits_{x \overset{<}{\to} -1} \dfrac{x^2 - x - 2}{x+1} = \lim\limits_{x \overset{<}{\to} -1} \dfrac{(x-2)(x+1)}{x+1} = \lim\limits_{x \overset{<}{\to} -1} (x$

$\Rightarrow$ we have $f_r'(-1) \neq f_l'(-1)$ then f does not differentiable at $x_0 = -1$

$2\bullet g(x) = \begin{cases} \dfrac{x}{1 + e^{1/x}} & \text{if } x \in \mathbb{R}^* \\ 0 & \text{if } x = 0 \end{cases}$ $\quad , x_0 = 0$

$g(x)$ is differentiable at $x_0 = 0$ iff $\lim\limits_{x \underset{<}{\overset{>}{\to}} 0} \dfrac{g(x) - g(0)}{x} = l$ finite

Then $\bullet \lim\limits_{x \overset{>}{\to} 0} \dfrac{g(x) - g(0)}{x} = \lim\limits_{x \overset{>}{\to} 0} \dfrac{\frac{x}{1 + e^{1/x}}}{x} = \lim\limits_{x \overset{>}{\to} 0} \dfrac{1}{1 + e^{1/x}} = 0 = g_r'(0)$ from which $g(x)$ is rigth derivable at $x_0 = 0$

$\bullet\ \bullet \lim\limits_{x \overset{<}{\to} 0} \dfrac{g(x) - g(0)}{x} = \lim\limits_{x \overset{<}{\to} 0} \dfrac{\frac{x}{1 + e^{1/x}}}{x} = \lim\limits_{x \overset{<}{\to} 0} \dfrac{1}{1 + e^{1/x}} = 1 = g_l'(0)$ from which $g(x)$ is left derivable at $x_0 = 0$

but $g_r'(0) \neq g_l'(0)$ so $g(x)$ does not derivable at $x_0 = 0$

Exercise 2

$\bullet$the function f is continuous on $\mathbb{R}^*$

Let us now study the continuity of f at $x_0 = 0$

$f(x)$ is continuous at $x_0 = 0$ iff $\lim\limits_{x\to 0} f(x) = f(0) = 0$?

$\lim\limits_{x\to 0} x^2 \sin\frac{1}{x} = 0 = f(0)$ $\left(\underset{x\to 0}{x^2 \nearrow 0} \text{ et } \left|\sin\frac{1}{x}\right| \le 1 \right)$ $\Leftrightarrow$ $f(x)$ is continuous at $x_0 = 0$ therefore f is continuous on $\mathbb{R}$

•The function f is differentiable on $\mathbb{R}^*$ because it is the product of functions differentiable on $\mathbb{R}^*$.

Let us now study the differentiability of f at $x_0 = 0$

We have $\lim\limits_{x\to 0} \dfrac{x^2 \sin\frac{1}{x} - 0}{x - 0} = \lim\limits_{x\to 0} x \sin\frac{1}{x} = 0 = f'(0)$,from where f is derivable at 0, then f is derivable on $\mathbb{R}$

we can writte

$$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

2. $f\,'$ is not continuous at $x_0 = 0$ because $\lim\limits_{x\to 0} \cos\frac{1}{x}$ does not exist *then* $\lim\limits_{x\to 0} 2x\sin\frac{1}{x} - \cos\frac{1}{x}$ does not exist

Exercise3

1.Calculate derivatives

$\triangleright f(x) = \ln(e^x) \Rightarrow \quad f'(x) = \dfrac{(e^x)}{e^x} = 1$

$\triangleright g(x) = \ln\left(\sin^2 x\right) \Rightarrow g'(x) = \dfrac{2\cos x \sin x}{\sin^2 x} = 2\dfrac{\cos x}{\sin x};$

$\triangleright h(x) = x + \sqrt{1+x^2} \Rightarrow h'(x) = 1 + \dfrac{2x}{2\sqrt{1+x^2}} = 1 + \dfrac{x}{\sqrt{1+x^2}}$

Show that : $\quad h'(x) = \dfrac{h(x)}{\sqrt{1+x^2}}$

We have $h'(x) = 1 + \dfrac{x}{\sqrt{1+x^2}} = \dfrac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}} = \dfrac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}} \Rightarrow h'(x) = \dfrac{h(x)}{\sqrt{1+x^2}}$

2.Calculate the nth derivative of

$f(x) = \sin\alpha x \Rightarrow f'(x) = \alpha\cos\alpha x = \alpha\sin\left(\alpha x + \frac{\pi}{2}\right)$

$\Rightarrow f^{(2)}(x) = \alpha^2\cos\left(\alpha x + \frac{\pi}{2}\right) = \alpha^2\sin\left(\alpha x + \frac{\pi}{2} + \frac{\pi}{2}\right) = \alpha^2\sin\left(\alpha x + 2\frac{\pi}{2}\right)$

$\Rightarrow f^{(3)}(x) = \alpha^3\cos\left(\alpha x + 2\frac{\pi}{2}\right) = \alpha^3\sin\left(\alpha x + 3\frac{\pi}{2}\right) ...$

$\Rightarrow \boxed{f^{(n)}(x) = \alpha^n\sin\left(\alpha x + n\frac{\pi}{2}\right) \quad \forall n\in\mathbb{N}}$

$g(x) = x^3\ln(1+x)$, we apply Leibniz's formula

Let $g(x) = f(x)\,h(x)$ then $g^{(n)}(x) = (f(x)\,h(x))^{(n)} = \sum\limits_{k=0}^{n} C_n^k f^{(k)}(x)\,h^{(n-k)}(x)$

such that $f(x) = x^3, \quad g(x) = \ln(1+x)$

We therefore look for the nth derivatives of $f(x)$ and $h(x)$

• $f(x) = x^3 \Rightarrow f'(x) = 3x^2 \Rightarrow f^{(2)}(x) = 6x \Rightarrow f^{(3)}(x) = 0$ then for $\forall n \geq 3$; we have $f^{(n)}(x) = 0$

•$h(x) = \ln(1+x) \Rightarrow h'(x) = \dfrac{1}{1+x} = (1+x)^{-1} \Rightarrow h^{(2)}(x) = (-1)(1+x)^{-2} \Rightarrow$ $h^{(3)}(x) = (-1)(-2)(1+x)^{-3}$

then for $\forall n \geq 2$; $h^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-(n+1)} = \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^{(n+1)}}$

hence

$g^{(n)}(x) = \sum_{k=0}^{n} C_n^k f^{(k)}(x) h^{(n-k)}(x) = C_n^0 f^{(0)}(x) h^{(n)}(x) + C_n^1 f^{(1)}(x) h^{(n-1)}(x) +$ $C_n^2 f^{(2)}(x) h^{(n-2)}(x) + C_n^3 f^{(3)}(x) h^{(n-3)}(x) + ... + C_n^n f^{(n)}(x) h^{(0)}(x)$

Which give

$g^{(n)}(x) = C_n^0 x^3 \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^{(n+1)}} + C_n^1 3x^2 \dfrac{(-1)^{n-2}(n-2)!}{(1+x)^{(n)}} + C_n^2 6x \dfrac{(-1)^{n-3}(n-3)!}{(1+x)^{(n-1)}} +$ $C_n^3 6 \dfrac{(-1)^{n-4}(n-4)!}{(1+x)^{(n-2)}}$

Exercise 4

Let $f(x) = e^{x^2}\cos x$

we have f is continuous and differentiable on $\mathbb{R}$, then f is continuous and differentiable on a part of $\mathbb{R}$

we apply Rolle's theorem for f on $[-\alpha, \alpha], \forall \alpha > 0$

As • f is continuous on $[-\alpha, \alpha]$

•f is differentiables on $]-\alpha, \alpha[$

and $f(-\alpha) = e^{(-\alpha)^2}\cos(-\alpha) = e^{\alpha^2}\cos(\alpha) = f(\alpha)$

then $\exists c \in]-\alpha, \alpha[$ such that $f'(c) = 0$

Exercise5

For $x = y$ the inegality is trivial.

1.For $x \neq y$. The function sin is continuous and differentiable on $\mathbb{R}$,dso we can apply the finite increase theorem for the function f on $[x, y]$ (if $x < y$ and $[y, x]$ if $y < x$) then $\exists c \in]x, y[$ such that $\sin y - \sin x = (y - x)(\sin c)' = (y - x)\cos c$

indeed $|\sin y - \sin x| = |(y - x)\cos c| = |y - x||\cos c|$,

as $|\cos c| \leq 1$, we obtain $|\sin y - \sin x| \leq |y - x| \Leftrightarrow |\sin x - \sin y| \leq |x - y|$.

2.the function $g(t) = \ln(1+t)$ is continuous and differentiable on $\mathbb{R}^*$therefore we can using the mean value theorem (on peut appliquer le théorème des accroissements finis) for g on $]0, x[$

then $\exists c \in]0, x[$ such that $\ln(1+x) - \ln 1 = (x - 0)(\ln(1+c))' = \dfrac{x}{1+c}$;

$c \in]0, x[\Rightarrow 0 < c < x \Leftrightarrow 1 < 1 + c < 1 + x \Leftrightarrow \dfrac{1}{1+x} < \dfrac{1}{1+c} < 1$,we muliply by x, $(x > 0)$

we find that $\dfrac{x}{x+1} < \dfrac{x}{1+c} < x$.we deduce that $\dfrac{x}{x+1} < \ln(1+x) < x$

Exercise6

Calculate the limit using the hospital rule

$\lim\limits_{x\to 0}\dfrac{\tan x - x}{x - \sin x} = \dfrac{0}{0}$ FI,we apply the hospital rule

$$\Rightarrow \lim_{x\to 0}\frac{\tan x - x}{x-\sin x} = \lim_{x\to 0}\frac{(\tan x - x)'}{(x-\sin x)'} = \lim_{x\to 0}\frac{\dfrac{1}{(\cos x)^2}-1}{1-\cos x}$$

$$\Rightarrow \lim_{x\to 0}\frac{\tan x - x}{x-\sin x} = \lim_{x\to 0}\frac{\dfrac{1-(\cos x)^2}{(\cos x)^2}}{1-\cos x} = \lim_{x\to 0}\frac{(1+\cos x)(1-\cos x)}{(\cos x)^2(1-\cos x)} = \lim_{x\to 0}\frac{1+\cos x}{(\cos x)^2} = 2$$

hence $\boxed{\lim\limits_{x\to 0}\dfrac{\tan x - x}{x-\sin x} = 2}$

$\lim\limits_{x\to 1}\dfrac{x^x-1}{\ln x - x + 1} = \lim\limits_{x\to 1}\dfrac{e^{x\ln x}-1}{\ln x - x + 1} = \dfrac{0}{0}$FI,we apply the hospital rule$\Rightarrow$

$$\lim_{x\to 1}\frac{\left(e^{x\ln x}-1\right)'}{(\ln x - x + 1)'} = \lim_{x\to 1}\frac{(\ln x + 1)e^{x\ln x}}{\dfrac{1}{x}-1}$$

$$\Rightarrow \lim_{x\to 1}\frac{x^x-1}{\ln x - x + 1} = \lim_{x\to 1}\frac{(\ln x + 1)e^{x\ln x}}{\dfrac{1}{x}-1} = \lim_{\substack{> \\ x\to 1 \\ <}}\frac{x(\ln x + 1)e^{x\ln x}}{1-x} = \frac{1}{0} = \pm\infty;$$

we obtain $\boxed{\lim\limits_{x\to 1}\dfrac{x^x-1}{\ln x - x + 1} = \pm\infty}$

$\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}} = 1^{+\infty}$ FI. we have $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}} = \lim\limits_{x\to 0} e^{\frac{1}{x^2}\ln(\cos x)} = e^{\lim \frac{1}{x^2}\ln(\cos x)}$

then $\lim\limits_{x\to 0}\frac{1}{x^2}\ln(\cos x) = \dfrac{0}{0}$ FI, we apply the hospital rule$\Rightarrow \lim\limits_{x\to 0}\frac{1}{x^2}\ln(\cos x) =$

$$\lim_{x\to 0}\frac{\dfrac{-\sin x}{\cos x}}{2x} = \lim_{x\to 0}\frac{\sin x}{x}^{\nearrow 1} \times \frac{-1}{2\cos x} = \frac{-1}{2}$$

Which give $\boxed{\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}} = e^{\frac{-1}{2}} = \dfrac{1}{\sqrt{e}}}$