Chapter 3

Work and Energy

III.1. Work

III.1.1. Elementary work

Consider a constant force $\vec{F}$ acting on a material point M. We define the elementary work dW of the force $\vec{F}$ by:

$dW = \vec{F} \cdot d\vec{l}$

$d\vec{l}$ is the elementary displacement.

$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$

$\overrightarrow{dl} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$dW = F_x.dx + F_y.dy + F_z.dz$

The work of a force $\vec{F}$ applied to a material point moving from point A to point B is:

$$W_{A\to B}(\vec{F}) = \int_A^B dW = \int_A^B \vec{F} \cdot \overrightarrow{dl}$$

The unit of work, in the SI system, is the **Joule**.

III.2. Constant force on a rectilinear movement:

Consider a material point M moving on the line segment [AB] under the effect of a force $\vec{F}$.

By definition, the work of the force $\vec{F}$on the rectilinear displacement AB is given by:

$$W_{A\to B}(\vec{F}) = \vec{F}.\overrightarrow{AB} = F.AB.\cos\alpha$$

α is the angle that $\vec{F}$ makes with $\overrightarrow{AB}$.

This work is

- positive (motor work) if the force is in the direction of movement (driving force):

$$\cos\alpha > 0 \ \Rightarrow 0 < \alpha < {}^{\pi}/_{2}$$

- Negative (resistant work) ifthe force is in the direction opposite to the displacement (force resisting):

$$\cos\alpha < 0 \ \Rightarrow {}^{\pi}/_{2} < \alpha < \pi$$

- zero if the force is perpendicular to the displacement:$\cos\alpha = 0 \ \Rightarrow \alpha = \frac{\pi}{2}$

III.3. Power

The power of a force $\vec{F}$ is the ratio of its work to the time taken to accomplish it.

Average power: $P_{moy} = \frac{W_{AB}}{\Delta t}$

Instantaneous power: $P(t) = \frac{dW_{AB}}{dt}$

The unit of power, in the SI system, is the Watt.

III.4. Kinetic energy

Let's calculate the work of the resultant of the force $\vec{F}$ applied to a material point of mass m between two points A and B.

$$W_{A\to B}(\vec{F}) = \int_A^B \vec{F} \cdot \overrightarrow{dl}$$

Now according to the fundamental principle of dynamics we have:

$$\vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt}$$

$$W_{A\to B}(\vec{F}) = \int_A^B m\frac{d\vec{V}}{dt} \cdot \overrightarrow{dl} = \int_A^B m \cdot d\vec{V} \cdot \frac{d\vec{l}}{dt} = \int_A^B m \cdot \vec{V}.d\vec{V}$$

Where $\frac{\overrightarrow{dl}}{dt} = \vec{V}$

However, since the displacement is very small, we can consider it as rectilinear, then the vectors are parallel. The work then becomes:

$$W_{A\to B}(\vec{F}) = \int_A^B m \cdot V.dV = m\int_A^B VdV = m\left[\frac{1}{2}V^2\right]_A^B = \frac{1}{2}mV_B^2 - \frac{1}{2}mV_A^2$$

The value $E_C = \frac{1}{2}mV^2$ is called the kinetic energy of the material point.

III.4.1. Kinetic energy theorem

“The work of the resultant of the forcesapplied to a material point between two points is equal to the variation of the kinetic energy of thematerial point »

$$W_{A\to B}(\vec{F}) = \Delta E_C = E_C(B) - E_C(A)$$

III.5. Potential Energy

III.5.1. Conservative force and non-conservative force

- A force is said to be conservative if its work between two points does not depend on the path followed, but only from the starting point and the ending point.
- Any conservative force derives from a potential function $E_p(x, y, z)$such that

$$\boldsymbol{\vec{F} = -\overrightarrow{grad}\,E_p(x, y, z)}$$

$$F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k} = \frac{\partial E_p}{\partial x}\vec{\imath} + \frac{\partial E_p}{\partial y}\vec{\jmath} + \frac{\partial E_p}{\partial z}\vec{k}$$

Examples: Force of gravity; force of weight; spring return force.

Remark:

If a force $\vec{F}$ is conservative, its rotational is zero:

$$\overrightarrow{rot}\,\vec{F} = \vec{\nabla} \wedge \vec{F} = \vec{0}$$

➢ The forces are called non-conservative when their work depends on the path followed.

III.6. Potential energy

✓ Potential energy is the potential function associated with the conservative force.

✓ Potential energy is the energy related to position.

✓ Potential energy is defined up to a constant; it is always referred to a reference frame taken as the origin to calculate it.

✓ The work of a conservative force is related to the potential energy by the expression:

$$W_{A\to B}(\vec{F}) = -\Delta E_P = E_P(A) - E_P(B)$$

III.7. Total Mechanical Energy

The mechanical (total) energy of a material point is the sum of kinetic and potential energies:

$$E_M = E_P + E_C$$

III.7.1. Principle of Conservation of Mechanical Energy

❖ The mechanical energy of a material point subjected to conservative forces is conserved.

$$E_M(A) = E_M(B) \Rightarrow E_P(A) + E_C(A) = E_P(B) + E_C(B) = cst$$

$$E_M(A) = cst$$

The variation in mechanical energy is zero

$$\Delta E_M = 0$$

❖ If one of the forces is not conservative, the mechanical energy is not conserved.

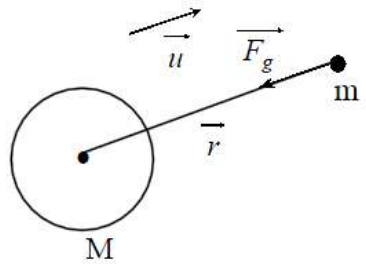
The variation of the mechanical energy between two points A and B is equal to the sum of the work of the non-conservative forces between these two points.

$$\Delta E_M = \sum_i W\,(\overrightarrow{F_{nc}})$$

Such that $\overrightarrow{F_{nc}}$ are the non-conservative forces

III.7.2. Examples of conservative forces

a) Force of gravity



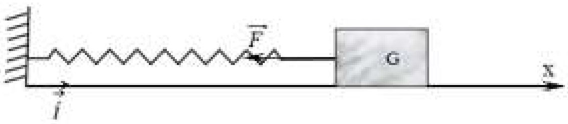
$\vec{F_g} = -G\frac{Mm}{r^2}\vec{u} \Rightarrow \vec{F_g} = -\overrightarrow{grad}E_P(r) = -\frac{dE_P}{dr}\vec{u}$

$G\frac{Mm}{r^2} = \frac{dE_P}{dr} \Rightarrow dE_P = G\frac{Mm}{r^2}dr$

$E_P(r) = \int G\frac{Mm}{r^2}dr$

$$E_P(r) = G\frac{Mm}{r} + cst$$

b) Elastic force



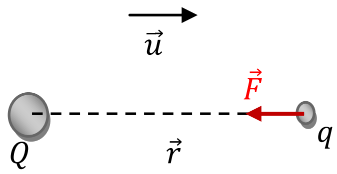
$\vec{F} = -kx\vec{i} \Rightarrow \vec{F} = -\overrightarrow{grad}E_P(r) = -\frac{dE_P}{dx}$

$dE_P = kxdx$

$$E_P(x) = \int kxdx = \frac{1}{2}kx^2 + cste$$

$$E_P(x) = \frac{1}{2}kx^2 + cst$$

c) Electric force



$\vec{F_e} = -K\frac{Qq}{r^2}\vec{u}$

Following the same reasoning as above, we will have:

$$E_P(r) = -K\frac{Qq}{r} + cst$$