Part II

3 Order in $\mathbb{R}$

3.1 upper and lower bounds

Definition 17 *Let A be non-empty set of $\mathbb{R}$, we say that*

› *The A is* ***bounded from above*** *if there exists $M \in R$ such that $x \leq M$ for all $x \in A$. We then say M is an* ***upper bound*** *of A*

› *The A is bounded from below if there exists $m \in R$ such that $m \leq x$ for all $x \in A$. We then say that m is alow bound of A.*
› *A is bounded if it is both* ***bounded above and bounded below****, that is $\exists (m, M) \in \mathbb{R}^2, \forall x \in A; \quad m \leq x \leq M$.*

▷*We say that A is unbounded if it is not bounded*

Example• $E = \{-2, -3, 0, 1, 5, 12\}$
-3 is a lower bound of E, because for all $x \in E$, $-3 \leq x$.
12 is an upper boubd of E, because for all $x \in E$, $x < 12$.
•$E =]-1, 3]$
$-2, -1$ are two lower bounds of E' , for all $x \in E$, $-2 < x$ and $-1 \leq x$.
3 and 4 are two upper bounds of $E', \forall\ x \in E$ such that $x \leq 3$ and $x < 4$

Remark 18 *The upper bound and lower bound of the set E are not unique. Indeed , $\mathbb{R}$ the set $E =]-1, 3]$ has infinite number of low bounds and upper bounds.*
• *$E = \mathbb{N}$ is bounded from below (each number $0 \leq x$ is lower bound) but not from above*

3.2 Maximum and Minimum

Let A be non-empty subset of $\mathbb{R}$,

Definition 19 • *The upper bound of A that belongs to A (necessarily unique) is called the maximum of the set A. It is denoted by* $\max(A)$. *In other words,*

$$M = \max(A) \Leftrightarrow \begin{cases} M \text{ is a upper bound of } A \\ M \text{ belonging to } A \end{cases}$$

If a set has a maximum, which is an upper bound for set and is actually the smallest of all possible upper bounds
The opposite is not true : a set can be bounded from above not admit a maximum

Definition 20 • *The lower bound of A that belongs to A (necessarily unique) is called the minimum of the set A. It is denoted by* $\min(A)$. *In other words,*

$$m = \min(A) \Leftrightarrow \begin{cases} m \text{ is a lower bound of } A \\ m \text{ belonging to } A \end{cases}$$

Additional notes

A bounded subset of $\mathbb{R}$ always has a maximum and a minimum.

Examples

The set $[0, 1]$ has a maximum of 1 and a minimum of 0.

The set $(-\infty, 0]$ has a maximum of 0, but no minimum.

The set $\mathbb{R}$ has no maximum or minimum.

3.3 Supremum and Infimum

Let A be a non-empty set of $\mathbb{R}$.

Definition 21 *Let $A \subset \mathbb{R}$ be bounded from above. The* ***supremum*** *or* ***least upper bound*** *of A is smallest of all upper bounds of A , which denoted by $M =$* $\sup(A)$

$$M = \sup(A) \Leftrightarrow \begin{cases} M \text{ is a upper bound of } A \\ M \text{ is the smallest of all upper bounds of } A \end{cases}$$

Definition 22 *Let $A \subset \mathbb{R}$ be bounded from above. The* ***infimum*** *or* ***greatest lower bound*** *of A is largest of all lower bounds of A , which denoted by $m =$* $\inf(A)$

$$m = \inf(A) \Leftrightarrow \begin{cases} m \text{ is a lower bound of } A \\ m \text{ is the largest of all lower bounds of } A \end{cases}$$

Additional notes

A bounded subset of $\mathbb{R}$ always has a maximum and a minimum.

An unbounded subset of $\mathbb{R}$ may or not may have a maximum or a minimum.

Example 23 *Let $A =]0, 1[$ the interval $]0, 1[$ is bounded and upper bound by 1and low bound by 0 $(\forall x \in A, \in 0 < x < 1)$.*

› The set of upper bound is [1, +∞[, which has the smallest of all upper bound is 1 . Therefore, ∃ sup(A) = 1, but $\max(A)$ does not exist $1 \notin A,$.

›]-∞, 0] is the set of lower bounds, which has the greatest lower bounds is 0 ∉ A. Therefore,∃ $\inf(A)$=0,but $\min(A)$ does not exist..

Remark 24 *the subset $] - \infty, 1]$ of $\mathbb{R}$ is not bouned from below.*

Remark 25 • *The supremum of a set is a more general concept than the maximum of a set. It is easy to show that if a set has a maximum, then this maximum is also the supremum of the set.*

For sake of simplicity:

• *If A has a maximum then $\sup(A) = \max(A)$.In the same way if $\sup(A) \in A$ then $\sup(A) = \max(A)$*

• *If A has a minimum then $\inf(A) = \min(A)$.In the same way if $\inf(A) \in A$ then $\inf(A) = \min(A)$.*

The following propositions are true:

Proposition 26 *:I) Any non-empty subset upper bounded (or lower bounded) of* $\mathbb{R}$ *has a* ***supremum*** *(less upper bound) (or i****nfimum (****greatest lower bound).*
II) The supremum and infinimum of a subset of $\mathbb{R}$*, if they exist, are unique*

Remark 27 ***: 1****) Property I, called the supremum property, is not true in the set of rational numbers* $\mathbb{Q}$*..*

2) Let A *be a non-empty subset of* $\mathbb{R}$*. We have* A *is bounded.* $\Rightarrow \forall x \in A$ *:* $\inf(A) \leq x \leq \sup(A)$.

Proposition 28 *Let* A *and* B *be two non-empty and bounded subsets of* $\mathbb{R}$ *and* $\blacksquare \in \mathbb{R}$*. We have the following properties:*

1. $A \subset B \quad \Longrightarrow \quad \sup A \leq \sup B$ *and* $\inf B \leq \inf A$*.*

2. $A \cup B$ *admits a finite upper bound and a finite lower bound, then:*

$$\sup(A \cup B) = \max\{\sup(A), \sup(B)\} \text{ and } \inf(A \cup B) = \min\{\inf(A), \inf(B)\}$$

Proposition 29 *3. if* $A \cap B \neq \varnothing$ *then* $\sup(A \cap B) \leq \min\{\sup(A), \sup(B)\}$

$$\text{and } \inf(A \cap B) \geq \max\{\inf(A), \inf(B)\}$$

Proposition 30 *4.* $\sup(A + B) = \sup(A) + \sup(B)$

$$\text{and } \inf(A + B) = \inf(A) + \inf(B)........$$

Proof. 1.a) $A \subset B \Leftrightarrow \forall x \in A \Rightarrow x \in B$ and in addition ,we have B is bounded $\Rightarrow \exists \sup(B), \exists \inf(B)$ so $\forall x \in A; \inf(B) \leq x \leq \sup(B)$:This prove that A is bounded above by $\sup(B)$ ($\sup(B)$ is an upper bound of A)and therefore $\sup(A)$ exists for all $x \in A$. then $x \leq \sup(A)$. Since $\sup(A)$ is the least upper bound of A we must have $\sup(A) \leq \sup(B)$

b)Similarly, we have $\forall x \in A \Rightarrow x \in B$ (B is bounded)

so $\forall x \in A; \inf(B) \leq x \Rightarrow \inf(B)$ is an lower bound of A but $\inf A$ is the greatest of all lower bounds of A

Therefore $\inf B \leq \inf A$. ■

4 Characterization of Supremum and infimum

Proposition 31 *let* $A \neq \varnothing$*;* $A \subset \mathbb{R}$*,if* A *has a* ***least upper bound****.* $M \in \mathbb{R}$ *so :*

$$M = \sup(A) = \begin{cases} \forall x \in A; x \leq M \\ \forall \varepsilon > 0; \exists\, x_0 \in A : M - \varepsilon < x_0 \leq M \end{cases}$$

if A *has an* ***greatest lower bound*** $m \in \mathbb{R}$ *then :*

$$m = \inf(A) = \begin{cases} \forall x \in A; m \leq x \\ \forall \varepsilon > 0; \exists\, x_0 \in A : m \leq x_0 < m + \varepsilon \end{cases}$$

4.1 Examples of training exercise

Example 32 *Let* $A = \left\{1 - \frac{1}{n}, n \in \mathbb{N}^*\right\}$ *Determine :* $\sup(A), \inf(A), \max(A), \min(A)$. *if they exist.*

Proof. for $n = 1 : 1 - \frac{1}{1} = 0 \in C \ \Rightarrow C$ is not empty.

therefore

$$A = \left\{1 - \frac{1}{1}, 1 - \frac{1}{2}, 1 - \frac{1}{3}, ...\right\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...\right\}.$$

so

$$\forall n \in \mathbb{N}^*, \quad 0 \leq 1 - \frac{1}{n} < 1 \ \Leftrightarrow \forall x \in A, \quad 0 \leq x < 1.$$

We have : $x \geq 0$ and $0 \in A$ so 0 is a lower bound for the set A, therefore $\min A = 0$ and so 0 is the smallest element in C

$$\Leftrightarrow \inf A = \min A = 0.$$

We show that $\sup A = 1$,using the charaterization of the supremum

$$\sup C = 1 \Leftrightarrow \begin{cases} \forall x \in A, x < 1 \\ \forall \varepsilon > 0, \exists x_0 \in A \text{ such that } 1 - \varepsilon < x_0 \end{cases}$$

therefore

$$\forall x_0 \in A \Rightarrow \exists n_0 \in \mathbb{N}^* \text{ such that } x_0 = 1 - \frac{1}{n_0}$$

this implies that

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}^* \text{ such that } 1 - \varepsilon < 1 - \frac{1}{n_0} \Rightarrow -\varepsilon < -\frac{1}{n_0}$$

this gives

$$\varepsilon \ > \frac{1}{n_0} \Longrightarrow \forall \varepsilon > 0, n_0 \varepsilon > 1.$$

by Archimedean property : $\exists n_0 \in \mathbb{N}^*$ such that $n_0 > \frac{1}{\varepsilon}$,

just take $n_0 = \left[\frac{1}{\varepsilon}\right] + 1$, thus $1 - \varepsilon < 1 - \frac{1}{n_0}$ then $\boxed{\sup A = 1}$. ■

Example 33 *Let* $B = \left\{\frac{1}{n^2} - 2, n \in \mathbb{N}^*\right\}$ *Determine* $\sup(B)$ *and* $\inf(B)$, *if they exist.*

Proof. We have $\forall n \in \mathbb{N}^*, \ 0 < \frac{1}{n^2} \leq 1$ then $\forall x \in D, \ -2 < x \ \leq -1 \Rightarrow B$ is bounded.

therefore $\exists \sup B, \exists \inf B$ such that $\forall x \in B, \quad \inf B \ < x \ \leq \sup B \Longrightarrow \inf B = -2$ and $\sup B = -1$

We have -1 $\in B$ so $\sup B = \max B = -1$,

We show that $\inf B = -2$,using the charaterization of the infimum

$$\inf B = -2 \Leftrightarrow \begin{cases} \forall x \in B, x > -2 \\ \forall \varepsilon > 0, \exists x_0 \in B \text{ such that } x_0 < -2 + \varepsilon \end{cases}, m = -2$$

therefore

$$x_0 \in B \Rightarrow \exists n_0 \in \mathbb{N}^* \text{ such that } x_0 = \frac{1}{n_0^2} - 2,$$

this implies that que

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}^* \text{ such that } \frac{1}{n_0^2} - 2 < -2 + \varepsilon \Rightarrow \varepsilon > \frac{1}{n_0^2}$$

this gives

$$\forall \varepsilon > 0, n_0^2 \varepsilon > 1.$$

By Archimedean property : $\exists n_0 \in \mathbb{N}^*$ such that $n_0 > \sqrt{\frac{1}{\varepsilon}}$,

just take : $n_0 = [\sqrt{\frac{1}{\varepsilon}}] + 1$, so $\frac{1}{n_0^2} < 0 + \varepsilon$ then $\inf B = 0$ ■

5 Completed real number line $\mathbb{R}$:

The completed real number line $\mathbb{R}$ is the set of all real numbers, together with two additional elements, $-\infty$ and $+\infty$.

The internal laws $(+, -)$ and the order relation $\leq$ are defined as follows :

• $\forall x \in \mathbb{R} \quad \begin{cases} x + (+\infty) = x + \infty = +\infty \\ x + (-\infty) = x - \infty = -\infty \end{cases}$

• $+\infty + (+\infty) = +\infty, \quad (-\infty) + (-\infty) = -\infty$

•$\forall x \in \mathbb{R}_+^* \quad \begin{cases} x \times (+\infty) = (+\infty) \times x = +\infty \\ x \times (-\infty) = (-\infty) \times x = -\infty \end{cases}$, • $\begin{cases} (+\infty) \times (+\infty) = (-\infty) \times (-\infty) = +\infty \\ (+\infty) \times (-\infty) = (-\infty) \times (+\infty) = -\infty \end{cases}$

•$\forall x \in \mathbb{R}, \quad -\infty < x < +\infty$ • $\begin{cases} +\infty \leq +\infty \\ -\infty \leq -\infty \end{cases}$

$\overline{\mathbb{R}}$ is called a completed real numbers line

Remark 34 *The operations of addition (+) and multiplication (×)arenotdefinedforallpairsofelementsinR.F* $(-\infty), (-\infty) + (\infty), (\infty) \times 0$, *and* $0 \times (\infty)$ *are undefined.*

6 Elements of topology in $\mathbb{R}$:

6.1 Open and closed sets in $\mathbb{R}$

Definition 35 *:1) An open set $O \subset \mathbb{R}$ is a set such that $\forall x \in O, \exists$ I an open interval $\subset \mathbb{R}$ such that $I \subset \mathbb{R}$ and I contains x.*

2) A closed set $F \subset \mathbb{R}$ is a set such that its complement $\mathbb{R} \backslash F$ is open.

Examples :1) the intervals : $]a,b[\,,]-\infty,a[\,,]a,+\infty[\,,]-\infty,+\infty[$ are all open sets.

2) the intervals : $[a,b]\,,[a,+\infty[\,,]-\infty,a]\,,]-\infty,+\infty[$ are all closed sets.

6.2 The neighborhood's notion of a point:

After the notion of an open set, the notion of neighborhood is very important in the study of convergence or limit.

Definition 36 *A subset V of $\mathbb{R}$ is a neighborhood of a point x_0 : $x_0 \in \mathbb{R} \Leftrightarrow \exists\alpha > 0;]x_0-\alpha, x_0+\alpha[\subset V$*

We write $V \in \upsilon(x_0)$.

$\rhd$ $]0,1[$ is a neighborhood of all of its elements., indeed : $\exists\alpha > 0;]x_0-\alpha, x_0+\alpha[$ with $x_0 \in]0,1[, \quad \alpha = \min(x_0-0, 1-x_0)$.
In the same way as the previous example we have : $]x_0-\alpha, x_0+\alpha[\subset]0,1[$.
$\rhd$The set $\mathbb{R}/\mathbb{Q}$ is not a neighborhood of any of its elements.; $\forall x_0 \in \mathbb{R}/\mathbb{Q}; \forall\alpha > 0$,the interval $]x_0-\alpha, x_0+\alpha[$ contains a rational number.

Conclusion:

The completed real number line $\mathbb{R}$ is a useful extension of the real number line that allows us to study concepts such as convergence and limit.