Part I

chap1: The field of real numbers

1 Subsets of $\mathbb{R}$

We recall the usual notations for the set of numbers :
$\mathbb{N} = \{0, 1, 2, \ldots..n\}$ is the set of natural numbers.
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$ is the set of relative integers.
$\mathbb{Q}$ is the set of rational numbers, or the set of fractions defined by
$\mathbb{Q} = \left\{ \frac{p}{q}, p \in \mathbb{Z} \text{ and } q \in \mathbb{N}^* \right\}$,for example : $3, \dfrac{2}{5}, \frac{-173}{1564}, \frac{4}{6} = \frac{2}{3}, \ldots.$
$D = \{r = \dfrac{p}{10^k} \in \mathbb{Q}$ such that $p \in \mathbb{Z}$ and $k \in \mathbb{N}\}$ is set of decimal numbers,
provide other examples: $3,135$ is written as $3135 \times 10^{-3} = \dfrac{3135}{10^3}$,
The set of real numbers is denoted $\mathbb{R}$ and we have the inclusions

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbf{D} \subset \mathbb{Q} \subset \mathbb{R}.$$

Proposition 1 *A real number is rational if and only if it has a decimal or periodic expansion starting at a certain rank.*

For example : $\frac{3}{5} = 0,6$, $\frac{1}{3} = 0,3333...$, $4,\underleftrightarrow{531}\underleftrightarrow{531}\underleftrightarrow{531}.....$

Definition 2 *A real number is irrational if and only if it is not rational. Thus, the set of irrational numbers is denoted* $\mathbb{R}/\mathbb{Q}$

Example : $\sqrt{2}$, π and exp, $\ln 2$, ..are those that are not rational, i.e., they cannot be written in the form $\dfrac{a}{b}$, $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^*$, or their decimal expansions are infinite and non-periodic

Proposition 3 $\sqrt{2}$ *is an irrational number* $\left(\sqrt{2} \notin \mathbb{Q}\right)$

Proof. Indeed let's reason by contradiction that $\sqrt{2} \in \mathbb{Q}$, i.e ,there exsit $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^*$ (a and b are relatively prime) , then $a^2 = 2b^2$. Thus a^2 is even, which implies a is even. There then exists $k \in \mathbb{Z}$ such that $a = 2k$, which gives $b^2 = 4k^2$ and also even . It follows that 2 divides a and b which contradicts the fact that a and b are relatively prime. Therefore $\sqrt{2}$ is an irrational number. ■

Remark 4 $\left\{ \begin{array}{l} \textit{rational} + \textit{irrational} = \textit{irrational} \\ \textit{rational} \times \textit{irrational} = \textit{irrational.} \end{array} \right.$ $(exercise)$

Remark 5 *On the other hand, the sum or product of two irrational numbers can be rational, for example:* $1 + \sqrt{2}$ *and* $1 - \sqrt{2}$.

2 Properties of $\mathbb{R}$

A field is a set $\mathbb{R}$ with two binary operations $(+) : \mathbb{R} \times \mathrm{R} \rightarrow \mathbb{R}$ and $\times : \mathbb{R} \times \mathrm{R} \rightarrow \mathbb{R}$,satisfying the following properties:
1) $x + y = y + x$ (commutative addition)
2) $x + (y + z) = (x + y) + z$ (assosiative addition)
3) there exist an element $0 \in \mathbb{R}$ such that $0 + x = x$ for all $x \in \mathbb{R}$ (identity element for +)
4)For every $x \in \mathbb{R}$, there exist an element $-x \in \mathbb{R}$ such that $x + (-x) = 0$ (additive inverse),
5) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (associative multiplication)
6) $x \cdot y = y \cdot x$ ($\cdot$ commutative)
7) There exists an element $1 \neq 0$ such that $1 \cdot x = x$ for all $x \in \mathbb{R}$ (identity element for multiplication)
8) For every element $x \neq 0$ in $\mathbb{R}$, there exists an element x^{-1} such that $x \cdot x^{-1} = 1$ (multiplicative inverse)
9)$x \cdot (y + z) = x \cdot y + x \cdot z$ (distributive property).

Example It is not hard to see that $\mathbb{N}$ and $\mathbb{Z}$ are not fields. In each case, what property of a field fails to hold?
but Both Q and R are fields.
Besides being fields, both $\mathbb{Q}$ and $\mathbb{R}$ are totally ordered sets. By totally ordered we mean that for any x, $y \in R$ either $x = y, x < y$, or $x > y$
We now present some very important rules of inequalities that we will use frequently in this course.

1) $x \leq y$ and $y \leq x \Rightarrow x = y$ (antisymmetry)
2) $x \leq y$ and $y \leq z \Rightarrow x \leq z$ (transitivity)
3) $\forall x \in \mathbb{R};\ x \leq x$ (reflexivity)
4) $x \leq y \Rightarrow x + z \leq y + z,\ z \in \mathbb{R}$
5) $(0 \leq x$ and $0 \leq y) \Rightarrow 0 \leq xy$

Consequences
1- From the relation "less than or equal" defined previously, we can define its symmetric relation "greater than or equal" in the same way, i.e.:
For all real numbers $x, y \in \mathbb{R}$, $x \geq y$ if and only if $y \leq x$.
2-We define the relation "strictly less than" $<$ by : For all $x, y \in \mathbb{R}$, $x < y$ if and only if $(x \leq y)$ and $(x \neq y)$. the relation "strictly greater than" $>$ by: : For all $x, y \in \mathbb{R}$, $x > y$ if and only if $(x \geq y)$ and $(x \neq y)$.

Exercise
show that $\forall a, b \in \mathbb{R}, a \leq b \Rightarrow -a \geq -b$ and $a.b = 0 \Leftrightarrow a = 0$ or $b = 0$
Solution
Since $(\mathbb{R}, +, \cdot)$ is a field, then $\forall a \in \mathbb{R}, a + (-a) = 0$ and $a + 0 = a$.
We can write this as follows:
$a + 0 \leq b + 0 \Leftrightarrow a + (b + (-b)) \leq b + (a + (-a))$ (+ is associative) $\Leftrightarrow (a + b) + (-b) \leq (a + b) + (-a)$

$$\Leftrightarrow \underbrace{(a+b)+(-(a+b))}_{0}+(-b) \leq \underbrace{(a+b)+(-(a+b))}_{0}+(-a) \text{ there fore, we}$$

have $-b \leq -a$ which implies $-a \geq -b$.
Proof of the second statement
If $a.b = 0 \Leftrightarrow a = 0$ or $b = 0$
We have $a.a^{-1} = 1$ and $\forall a \in \mathbb{R}$; $a.0 = 0$ and $1.a = a$ so $a.b = 0 \Rightarrow \underbrace{(a^{-1}.a)}_{1}.b =$

$a^{-}.0 = 0 \Rightarrow b = 0$
and the same for $a \Rightarrow a.b = 0 \Rightarrow a.\underbrace{(b.b^{-1})}_{1} = 0.b^{-1} = 0 \Rightarrow a = 0$

Proposition 6 ***:Archimedean property.***

$$\forall x, y \in \mathbb{R}, \text{ and } y > 0; \quad \exists n \in \mathbb{N}^* \text{such that } \ x < ny.$$

$\mathbb{R}$ is said to be Archimedean.
As a consequence of the Archimedean axiom, $\boxed{n \leq x < n+1}$

2.1 The integer part

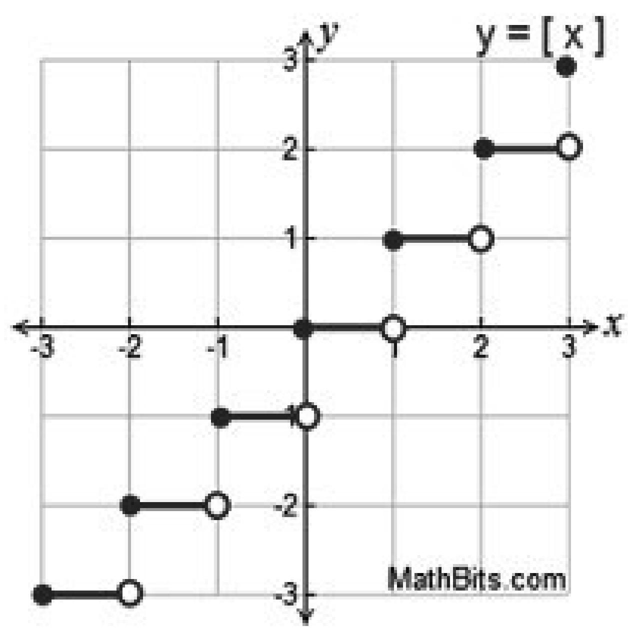
Definition 7 *Let x be a real number, there exists a unique integer $\in \mathbb{Z}$ denote by $int(x)$ (or $[x], E(x)$),such that*

$$[x] \leq x < [x]+1$$

This is the greatest integer less than or equal to x. We call $[x]$ the integer part of x. (or the floor function)

Example 8 $[2.30] = 2$, $[\sqrt{3}] = 1$, $E[-2, 14] = -3$.

Graphical representation of the integer part function $f(x) = int(x) = [x]$



2.1.1 Properties of the interger part

1) The integer part is an increasing function.

2) $\forall x \in \mathbb{R},\ x = [x] = int\,(x) \Longleftrightarrow x \in \mathbb{Z}$

3)$\forall\,(x, n) \in (\mathbb{R} \times \mathbb{Z})\,, int\ (x + n) = int\,(x) + n.$

Remark 9 *In general,* $[x + n] \neq [x] + [n]$ *and* $[n \cdot x] \neq n \cdot [x]$

Remark 10 $\bullet$*The integer part in increasing i.e* $\forall\,(x, y) \in \mathbb{R}^2,\ x \leq y \Longrightarrow [x] \leq [y]$

$\bullet$ How wever, the integer part is not strictly increasing;for example , $0 < \frac{2}{3}$ but $\left[\frac{2}{3}\right] = 0.$

Remark 11 *if we have* $n \in \mathbb{Z}$ *such that* $n = [x]$ *then* $n = [x] \Longleftrightarrow n \leq x < n + 1 \Longleftrightarrow n - 1 < x \leq n.$

In fact, $[x]$ is odd if n is an integer number. $[x] = -[-x]$

2.2 Density of $\mathbb{Q}$ and $\mathbb{R}/\mathbb{Q}$ in $\mathbb{R}$

Theorem 12 *for any two real numbers* x *and* y *, there existes a rational number* r *such that*

$$\forall x, y \in IR,\ x < y \Longrightarrow \exists r \in \mathbb{Q} : x < r < y.$$

Proof. Let x and y be two real numbers such that $x < y$, *and* let $z = y - x > 0$,

Since $\mathbb{R}$ is Archimedean, $\Longrightarrow \exists n \in \mathbb{N}^*$ such that $nz > 1 \Longrightarrow z > \dfrac{1}{n}.$

We have

$$[nx] \leq nx < [nx] + 1$$

Let $m = [nx] + 1$,then

$$m - 1 \leq nx < m \Longrightarrow \frac{m-1}{n} \leq x < \frac{m}{n} < x + \frac{1}{n} < x + z = y$$

therefore,

$$\forall x, y \in \mathbb{R},\quad x < y \text{ on a } x < \frac{m}{n} < y.$$

■

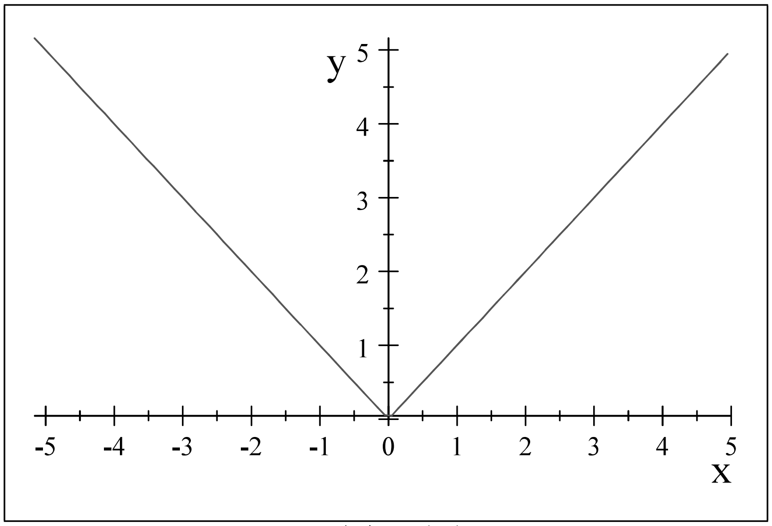
Theorem 13 *between any two real numbers* x *and* y*, there exists an irrational number.* $(Exercise)$

2.3 Absolute value

Definition 14 *Let $x \in \mathbb{R}$, The absolute value of x (denoted by $|x|$),is the real number defined as follows:*

$$|x| = \begin{cases} x & si \quad x > 0 \\ 0 & si \quad x = 0 \\ -x & si \quad x < 0 \end{cases}, or \qquad |x| = \max(-x, x).$$

Graphical representation of $f(x) = |x|$



$f(x) = |x|$

2.3.1 Properties of the absolute value :

The absolute value function satisfies the following properties :

1) $\forall x \in \mathbb{R},\ |x| \geq 0$ and $-|x| \leq x \leq |x|$

2) $\forall x \in \mathbb{R}, \sqrt{x^2} = |x| \quad \left(|x|^2 = x^2\right)$

3)$\forall a \geq 0, \forall x \in \mathbb{R};\ |x| < a \Leftrightarrow -a \leq x \leq +a$

4) $\forall x, y \in \mathbb{R},\ |x \cdot y| = |x| \cdot |y|$

5) $\forall x, y \in \mathbb{R}, |x + y| \leq |x| + |y|$ (first triangel inequality)

6) $\forall x, y \in \mathbb{R}, ||x| - |y|| \leq |x + y|$ (second triangel inequality) (Exercise)

Proof. a) proof of the first triangle inequality

$$\forall x, y \in \mathbb{R}, \quad |x + y| \leq |x| + |y|$$

we have

$$\begin{cases} -|x| \leq x \leq |x| & \forall x \in \mathbb{R}........(1) \\ \\ -|y| \leq y \leq |y| & \forall y \in \mathbb{R}........(2) \end{cases}$$

$(1) + (2)$ we have

$$-|x| - |y| \leq x + y \leq |x| + |y|,$$

or

$$-(|x| + |y|) \leq x + y \leq (|x| + |y|),$$

since $|x+y| \leq |x| + |y|$.

■

Exercise . Let $f(x) = x^2 - 3x + 4$ for $x \in [-1, 1]$. Find a number $M > 0$ such that $|f(x)| \leq M$ for all $-1 \leq x \leq 1$.

Solution. Clearly, if $-1 \leq x \leq 1$ then $|x| \leq 1$. Apply the triangle inequality and the properties of the absolute value:

$$|f(x)| = |x^2 - 3x + 4| \leq |x^2| + |3x| + |4| = |x|^2 + 3|x| + 4$$
$$\leq (1)^2 + 3(1) + 4 = 8$$

Therefore, if $M = 8$ then $|f(x)| \leq M$ for $x \in [-1, 1]$

2.4 Intervals

Definition 15 *Let I be a subset of $\mathbb{R}$, I is an interval of $\mathbb{R}$ if $\forall a, b \in I : a < b$ and $\forall x \in \mathbb{R}$; $a \leq x \leq b \Rightarrow x \in I$.*

Remark 16 *$I = \varnothing$ is an interval.*

$I = \mathbb{R}$ is also an interval.

Types of intervals

There are a total of 9 possible types of intervals, of which 4 are bounded and 5 are unbounded.Types of intervals

i) **Bounded intervals**: Let $a, b \in \mathbb{R}$

- $[a, b] = \{x \in \mathbb{R};\ a \leq x \leq b\}$ is a closed interval.
- $]a, b[= \{x \in \mathbb{R};\ a < x < b\}$ is an open interval.
- $[a, b[= \{x \in \mathbb{R};\ a \leq x < b\}$ is an open interval on b.
- $]a, b] = \{x \in \mathbb{R};\ a \leq x < b\}$ is an open interval on a.

ii) **Unbounded intervals :**

- $]-\infty, a[$, $]-\infty, a]$ is an unbounded interval on the right.
- $]b, +\infty[$; $[b, +\infty[$ is an unbounded interval on the left.
- $]-\infty, +\infty[$.

Notes :

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$, $\mathbb{R}_+^* =]0, +\infty[$, $\mathbb{R}_-^* =]-\infty, 0[$, $\mathbb{R}_- =]-\infty, 0]$ and $\mathbb{R}^* =]-\infty, 0[\cup]0, +\infty[$.

Examples

$[-1, 1]$ is a bounded closed interval.

$]1, 3[$ is a bounded open interval.

$[-2, +\infty)$ is an unbounded interval on the right.

$(-\infty, 1]$ is an unbounded interval on the left.