Mathematical Logic
Series of tutorials N°03 : Propositional logic

Exercise 1
1. Let A be the following proposition : « All men are bearded ». Check the correct formulations of proposition $\neg A$.

☐ « Not all men are bearded.»
☐ « No man is bearded.»
☐ « There exists a man who is not bearded.»
☐ « There is at least one man who is not bearded.»
☐ « There is only one man who is not bearded.»

2. Here is a list of simple propositions A and B, the truth values of which you know. In each case, express the truth value of proposition $A \wedge B$.

T F
☐ ☐ A : « Paris is the capital of France.» and B : « 1+1 = 2».
☐ ☐ A : « A cat has five legs.» and B : « A square has four sides.».
☐ ☐ A : « A right-angled triangle has a right angle.» and B : « Two parallel lines intersect at a point.».
☐ ☐ A : « 3 * 8 = 32» and B : « Paris is the capital of France».
☐ ☐ A : « Berlin is the capital of Spain.» and B : « A right-angled triangle has three equal sides.».
☐ ☐ A : « A fly can fly.» and B : « Canada is a country in the North American continent.».

3.Here is a list of simple propositions A and B, the truth values of which you know. In each case, express the truth value of proposition $A \vee B$.

T F
☐ ☐ A : « Paris is the capital of France.» and B : « 1+1 = 2».
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Exercise 2 Are the following expressions a well-formed formulas ?
1. $p \wedge \neg q$, 2. $p \vee \vee r$, 3. $(p \vee (\neg p))$, 4. $(p \vee \neg p)$.

Exercise 3 Give the truth tables for the following formulas, and indicate the equivalences between these formulas.
1. $\neg(p \wedge q)$, 2. $\neg p \vee \neg q$, 3. $\neg(p \vee q)$, 4. $\neg p \wedge \neg q$, 5. $p \vee (p \wedge q)$, 6. $p \wedge (p \vee q)$, 7. p

Exercise 4 Let p and q be two propositional variables denoting respectively« 'It is cold » and « 'It is raining ». Write a simple sentence corresponding to each of the following statements :
1. $\neg p$, 2. $p \wedge q$, 3. $p \vee q$, 4. $q \vee \neg p$, 5. $\neg p \wedge \neg q$, 6. $\neg\neg q$

Exercise 5 transform the following sentences into propositional logic formulas by specifying the universe of the discourse :
1. This engine is not noisy, but it consumes a lot.
2. it is not true that Peter will come if Mary or John come.
3. John is not only stupid, but he is also evil.
4. I go to the beach or to the cinema on foot or by car.
5. Peter has no brothers or sisters, but he has a cousin.
6. If it's raining and sunny then there's a rainbow.
7. John will only go to the cinema if he has finished his homework.

Exercise 6 Enigma : Three colleagues, Ahmed, Ali, and Mostafa, have lunch together every working day. The following statements are true :

1. If Ahmed orders a dessert, Ali orders one to,
2. Every day, either Mostafa or Ali, but not both, orders a dessert,
3. Every day, either Ahmed or Mostafa, or both, order a dessert,
4. If Mostafa orders a dessert, Ahmed does the same.

Questions :

1. Express the data of the problem as propositional formulas.
2. What can be deduced about who orders a dessert ?
3. Can we reach the same conclusion by removing one of the four statements ?

Exercise 7 Sheffer's connector is defined, denoted as | (Sheffer bar) which is the NAND by p| q $\equiv \neg(p \wedge q)$.

1. Give the truth table for the formula (p | q).
2. Give the truth table for the formula ((p | q)|(p | q)).
3. Express the connectors $\neg$, $\vee$ and $\rightarrow$ by using the Sheffer bar.

Exercise 8 Establish the truth tables for the following formulas and determine if they are valid, satisfiable, or unsatisfiable.

a. $(\neg P \wedge \neg Q) \rightarrow (\neg P \vee R)$
b. $P \wedge (Q \rightarrow P) \rightarrow P$
c. $(P \vee Q) \wedge \neg P \wedge \neg Q$
d. $(P \rightarrow Q) \wedge (Q \vee R) \wedge P$
e. $((P \vee Q) \rightarrow R) \leftrightarrow P$

Exercise 9 Find the disjunctive normal forms :
a. $(A \vee B \vee C) \wedge (C \vee \neg A)$
b. $(A \vee B) \wedge (C \vee D)$
c. $\neg((A \vee B) \rightarrow C)$

Exercise 10 Find the conjunctive normal forms :
a. $(A \vee B) \rightarrow (C \wedge D)$
b. $(A \vee (\neg B \wedge (C \vee (\neg D \wedge E))))$
c. $A \leftrightarrow (B \wedge \neg C)$

Exercise 11 Prove that the following formulas are theorems :
a. $\vdash A \leftrightarrow A$,Knowing that $A \rightarrow A$ should not be taken as an axiom.
b. $\vdash \neg B \rightarrow (B \rightarrow A)$

Exercise 12 Establish the following deductions :
a. $A \rightarrow (B \rightarrow C), A \wedge B \vdash C$
b. $A \rightarrow (B \rightarrow C), B \vdash A \rightarrow C$
c. $A, B \wedge C, A \wedge C \rightarrow E \vdash E$
d. $E, E \rightarrow (A \wedge D), D \vee F \rightarrow G \vdash G$

The axioms of propositional logic are :

- **1a.** $(A \rightarrow (B \rightarrow A))$
- **1b.** $(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C))$
- **1c.** $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
- **1d.** $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
- **2.** $A \rightarrow (B \rightarrow A \wedge B)$
- **3a.** $A \wedge B \rightarrow A$
- **3b.** $A \wedge B \rightarrow B$
- **4a.** $A \rightarrow A \vee B$
- **4b.** $B \rightarrow A \vee B$
- **5.** $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$
- **6.** $B \rightarrow ((B \rightarrow C) \rightarrow C)$
- **7.** $A \rightarrow A$
- **8.** $(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$
- **9.** $(A \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow (A \leftrightarrow B))$