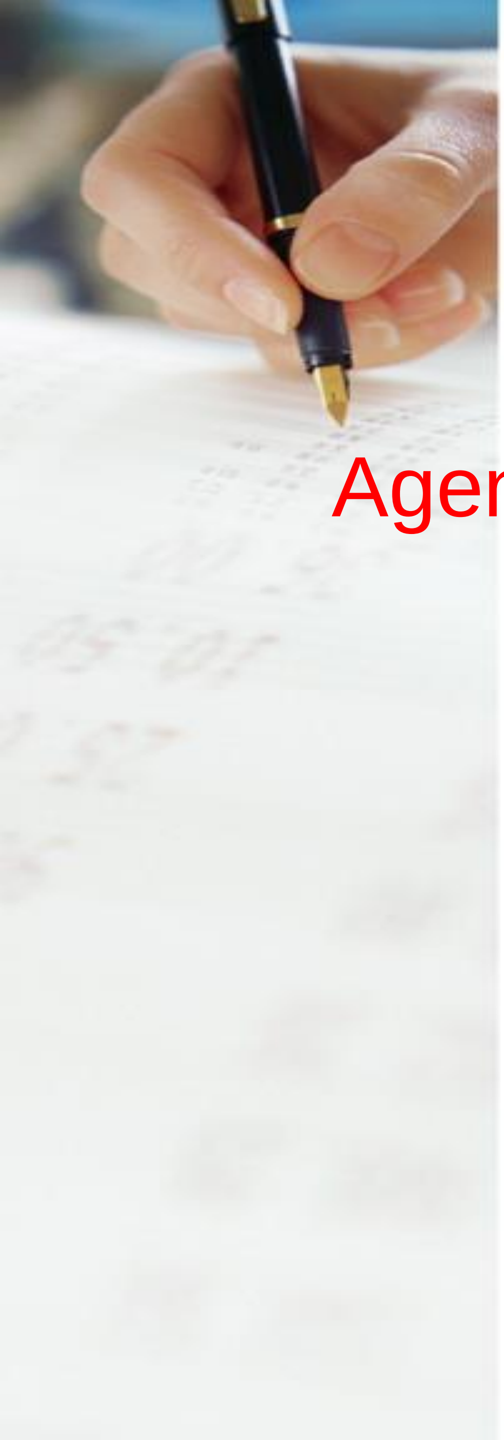
BADJI MOKHTAR ANNABA UNIVERSITY
FACULTY OF TECHNOLOGY
Computer Sciences and Electronics Department
1ST YEAR

Teaching mode : online


KINEMATICS


Spacial motion



Agendas

01 Motion in cylindrical Coordinates

02 Motion in spherical coordinates

Cylendrical Coordinates

By definition, the cylindrical coordinates of a point M are:

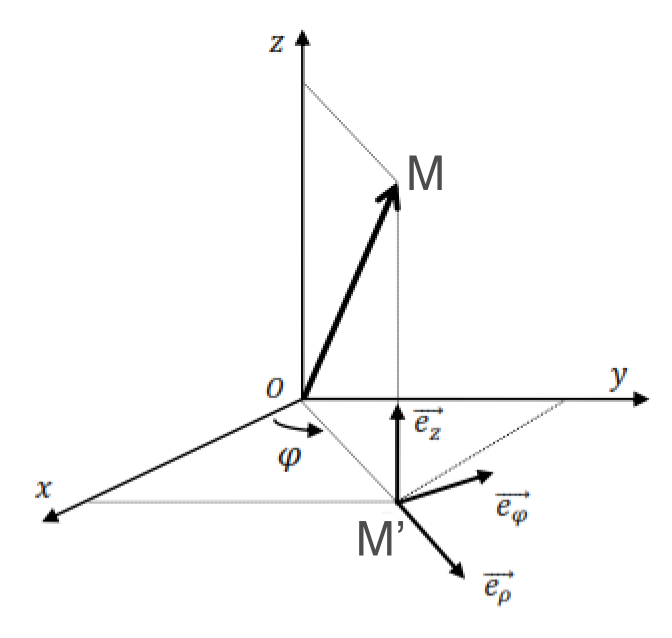
(r, φ, *z*), and unit basis vectors $(\overrightarrow{e_p}, \overrightarrow{e_\varphi}, \overrightarrow{k})$ *where* $r \geq 0$ *and* $0 \leq \theta \leq 2\pi$

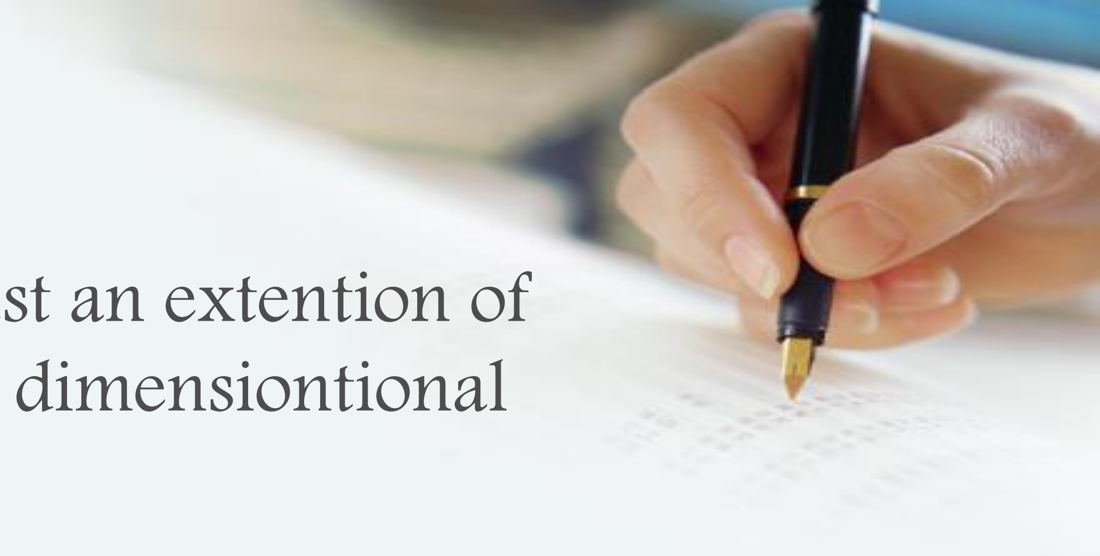
$r \geq 0$

$0 \leq \varphi \leq 2\pi$

$-\infty \leq z \leq \infty$

$$\begin{cases} \overrightarrow{e_p} = \cos\varphi \vec{i} + \sin\varphi \vec{j} \\ \overrightarrow{e_\varphi} = -\sin\varphi \vec{i} + \cos\varphi \vec{j} \\ \overrightarrow{e_z} = \vec{k} \end{cases}$$





- The cylendrical Coordinates are just an extention of the polar coordinates for the three dimensiontional coordinates system

- The cylendrical coordinates of a point M are (r,φ,Z)

  * r is the distance between the point P and the Z axe

 *φ is the angle between the axis(x) and the vector OM' where M' is the projection of M on the (xy) plane

* Z is the height of the point M

- Convert Cylendrical to Cartesian:

$$\begin{pmatrix} r \\ \varphi \\ z \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{cases} x = r.cos\varphi \\ y = r.sin\varphi \\ z = z \end{cases}$$

- Convert Cartesian to Cylendrical:

- $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} r \\ \varphi \\ z \end{pmatrix} : \begin{cases} r = \sqrt{x^2 + y^2} \\ \mathrm{tg}(\varphi) = \frac{y}{x} \\ z = z \end{cases}$$

- **Position vector:**

$$\overrightarrow{OM} = \overrightarrow{\mathrm{OM}'} + \overrightarrow{\mathrm{M'\,M}} = r.\overrightarrow{e_p} + z\,\overrightarrow{e_z}$$

$$\overrightarrow{\mathrm{O}\mathcal{M}} = \mathrm{r}\cos\varphi.\vec{\mathrm{i}} + \mathrm{r}\sin\varphi.\vec{\mathrm{j}} + Z\vec{\mathrm{k}}$$

The elementary displacement is given by the expression

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\boldsymbol{\theta}^2 + \mathrm{d}z^2$$

- **Velocity vector:**

- $$\vec{v}(t) = \frac{d\overrightarrow{OM}}{dt} = \frac{d}{dt}\left(p.\overrightarrow{e_p} + z\overrightarrow{e_z}\right) =$$

$$\vec{v}(t) = \frac{d}{dt}\left(p.\overrightarrow{e_p}\right) + \frac{d}{dt}\left(z\overrightarrow{e_z}\right)$$

$$= \dot{p}\ \overrightarrow{e_p} + \mathrm{p}\frac{d\overrightarrow{e_p}}{dt} + \dot{Z}\ \overrightarrow{e_z} + \mathrm{Z}\ \frac{d\overrightarrow{e_z}}{dt}$$

$$\overrightarrow{e_p} = cos\varphi\vec{\imath} + sin\varphi\vec{\jmath}$$

$$\frac{d\overrightarrow{e_p}}{dt} = -\dot{\varphi}sin\varphi\vec{\imath} + \dot{\varphi}cos\varphi\vec{\jmath}\ = \dot{\varphi}\ \overrightarrow{e_\varphi}$$

$$\vec{v}(t) = \dot{p}\ \overrightarrow{e_p} + \mathrm{p}\ \dot{\varphi}\ \overrightarrow{e_\varphi} + \dot{Z}\ \overrightarrow{e_z}$$

Magnitude

$$|\vec{v}| = \sqrt{{v_r}^2 + {v_\varphi}^2 + {v_z}^2} \quad \text{with}$$

$$\vec{v}\begin{cases} v_r = \dot{p} & \textit{radial} \\ v_\varphi = p\dot{\varphi} & \textit{transversal} \\ v_z = \dot{z} & \textit{azimutal} \end{cases}$$

- **Acceleration vector:**

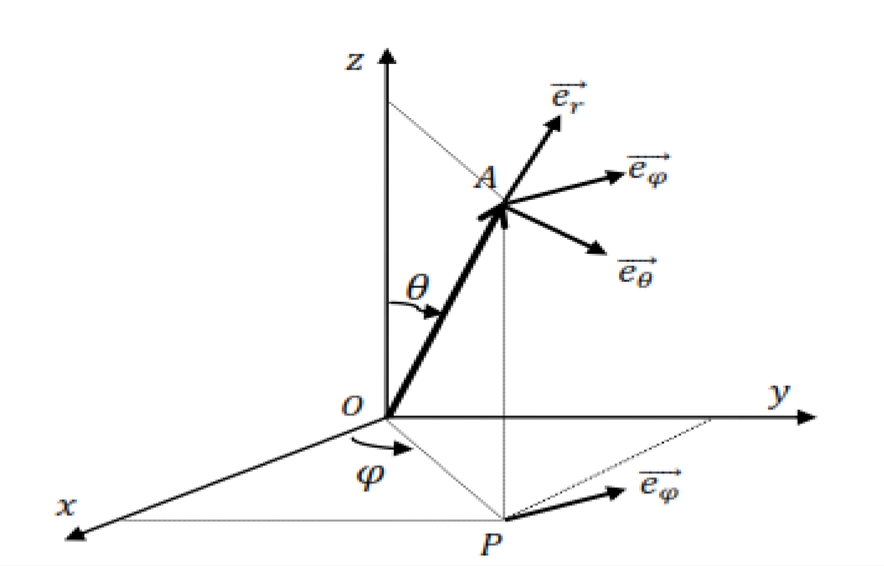
- $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{p}\,\overrightarrow{e_p} + p\,\dot{\varphi}\,\overrightarrow{e_\varphi} + \dot{Z}\,\overrightarrow{e_z}\right)$
- $\vec{a} = \ddot{p}\,\overrightarrow{e_p} + \dot{p}\,\frac{d\overrightarrow{e_p}}{dt} + \ddot{\varphi}\,p\,\overrightarrow{e_\varphi} + \dot{\varphi}\,\dot{p}\,\overrightarrow{e_\varphi} + p\,\dot{\varphi}\,\frac{d\overrightarrow{e_\varphi}}{dt} + \ddot{Z}\,\overrightarrow{e_z}$
- $\overrightarrow{e_\varphi} = -sin\varphi\vec{i} + cos\varphi\vec{j}$
- $\frac{d\overrightarrow{e_\varphi}}{dt} = -\,\dot{\varphi}\,cos\varphi\,\vec{i} - \dot{\varphi}\,sin\varphi\,\vec{j} = \dot{\varphi}\,(cos\varphi\vec{i} + sin\varphi\vec{j})$
- $\frac{d\overrightarrow{e_\varphi}}{dt} = \dot{\varphi}\,\overrightarrow{e_p}$
- $\vec{a} = (\ddot{p} - \dot{\varphi}^2 p)\,\overrightarrow{e_p} + (2\,\dot{p}\,\dot{\varphi} + \ddot{\varphi}\,p\,)\,\overrightarrow{e_\varphi} + \ddot{Z}\,\overrightarrow{e_z}$

Magnitude

$$|\vec{a}| = \sqrt{{a_r}^2 + {a_\varphi}^2 + {a_z}^2} \quad \text{with}$$

$$\vec{a} \begin{cases} a_r = \ddot{p} - \dot{\varphi}^2 p & \textit{radial} \\ a_\varphi = 2\,\dot{p}\,\dot{\varphi} + \ddot{\varphi}\,\mathrm{p} & \textit{transveral} \\ a_z = \ddot{z} & \textit{azimutal} \end{cases}$$

Spherical Coordinates (r, θ, ϕ)



$$0 \leq r \leq \infty$$
$$0 \leq \phi \leq \pi$$
$$0 \leq \theta < 2\pi$$

$$\begin{cases} x = \mathrm{R}\ \sin\theta \cos\varphi \\ y = \mathrm{R}\ \sin\theta\, sin\varphi \\ z = R.cos\varphi \end{cases}$$

- The spherical coordinates of a point P are (r, ϕ, θ) where:

R : is the distance between the origin and the point M

ϕ:is the angle between the axis(x) and the point m, where m is the projection of point M on the (xy) plane

θ : is the angle between vector R and Z axe

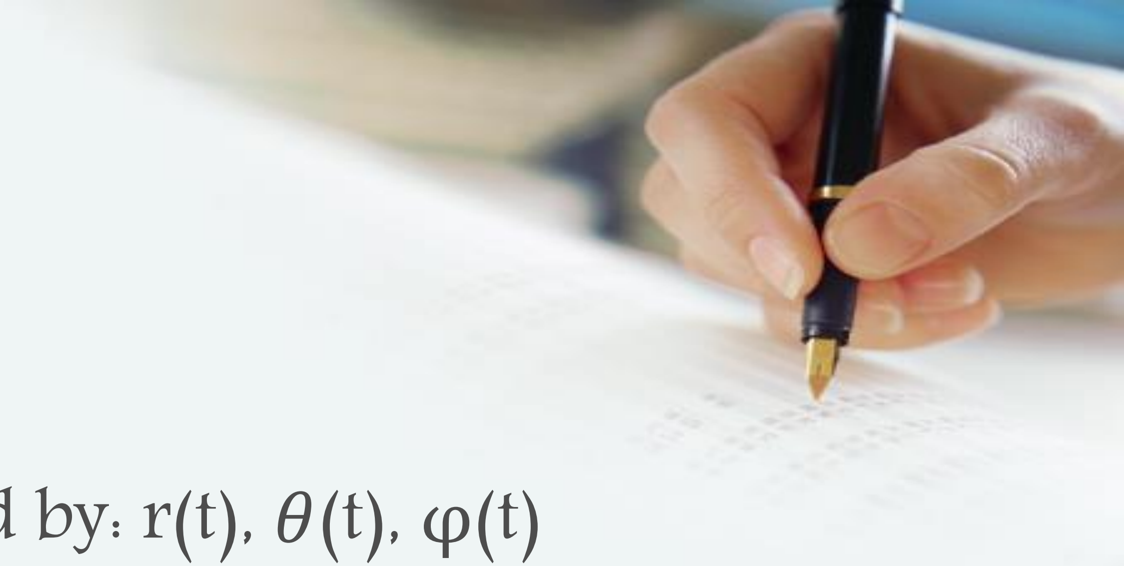
Convert from spherical to cartesian :

$$\begin{cases} x = r.cos\theta \\ y = r.sin\theta \\ z = R.cos\varphi \end{cases} \qquad r = R\ sin\varphi$$

$$\begin{pmatrix} R \\ \theta \\ \varphi \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{cases} x = R.sin\varphi.cos\theta \\ y = R.sin\varphi.sin\theta \\ z = R.cos\varphi \end{cases}$$

- Convert Cartesian to spherical:

- $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} R \\ \theta \\ \varphi \end{pmatrix} : \begin{cases} R = \sqrt{x^2 + y^2 + z^2} \\ cos\varphi = \frac{z}{R} \\ tg\theta = \frac{y}{x} \end{cases}$



The position vector $\overrightarrow{OM}$ is defined by: r(t), θ(t), φ(t)

Using relationships between vectors $(\overrightarrow{e_r}, \overrightarrow{e_\varphi}\,\overrightarrow{e_\theta})$

$$\begin{cases} \overrightarrow{e_r} = sin\theta\, cos\varphi\vec{i} + sin\theta\,.\,sin\varphi\vec{j} + \; cos\theta\,\vec{k} \\ \overrightarrow{e_\varphi} = -sin\varphi\vec{i} + cos\varphi\vec{j} \\ \overrightarrow{e_\theta} = cos\theta cos\varphi\vec{i} + cos\theta\, sin\varphi\vec{j} - sin\theta\vec{k} \end{cases}$$

- **Position vector:**
- $R = \|\overrightarrow{OM}\| = R\,\overrightarrow{e_r}$

The position vector $\overrightarrow{OM}$ is defined by: $R(t)$, $\theta(t)$, $\phi(t)$

- $\overrightarrow{OM} = R[\sin\theta\cos\varphi\vec{i} + \sin\theta\,.\sin\varphi\vec{j} + \cos\theta\,\vec{k}] = R\overrightarrow{e_r}$

- **Velocity vector:**

$$\vec{v}(t) = \frac{d\overrightarrow{OM}}{dt} = \frac{d}{dt}(R\overrightarrow{e_r}) = \frac{dR}{dt}\overrightarrow{e_r} + R\frac{d\overrightarrow{e_r}}{dt}$$

- $\frac{d\overrightarrow{e_r}}{dt} = \dot{\theta}\left[cos\theta cos\varphi\vec{i} + cos\theta sin\varphi\vec{j} - sin\theta\vec{k}\right] + \dot{\varphi}sin\theta[-sin\varphi\vec{i} + cos\varphi\vec{j}]$

- $\frac{d\overrightarrow{e_r}}{dt} = \dot{\theta}\overrightarrow{e_\theta} + \dot{\varphi}sin\theta\overrightarrow{e_\varphi}$

Substituting into the equation of speed, we obtain:

- $\vec{v}(t) = \dot{R}\,\overrightarrow{u_R} + R\dot{\theta}\overrightarrow{e_\theta} + R\,sin\theta\,\dot{\varphi}\,\overrightarrow{e_\varphi}$

Then the three components of the velocity vector appear as follows:

- $\vec{v}(t) = \overrightarrow{v_R} + \overrightarrow{v_\theta} + \overrightarrow{v_\varphi}$
- $\vec{v}(t) = \frac{dR}{dt}\overrightarrow{u_R} + R\frac{d\theta}{dt}sin\varphi\overrightarrow{u_\theta} + R\frac{d\varphi}{dt}\overrightarrow{u_\varphi}$

Magnitude

$$|\vec{v}| = \sqrt{{v_R}^2 + {v_\theta}^2 + {v_\varphi}^2} \quad \text{withe}$$

$$\vec{v}\begin{cases} v_R = \dot{R} & radial \\ v_\theta = R\dot{\theta} & azmutal \\ v_\varphi = \mathrm{R}\, sin\theta\, \dot{\varphi} & transversal \end{cases}$$

Acceleration vector:

- $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{R}\,\overrightarrow{u_R} + R\dot{\theta}\overrightarrow{e_\theta} + R\,\sin\theta\;\dot{\varphi}\;\overrightarrow{e_\varphi}\right)$
- $\vec{a} = \left(\ddot{R} - R\dot{\theta}^2 - R\dot{\varphi}^2\right)\overrightarrow{e_r} + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta} - R\dot{\varphi}^2\sin\theta\,\cos\theta\right)\overrightarrow{e_\theta}$

$+ \left(R\ddot{\varphi}\sin\theta + 2\dot{R}\,\dot{\varphi}\sin\theta + 2R\,\dot{\varphi}\,\dot{\theta}\,\cos\theta\right)\overrightarrow{e_\varphi}$

Magnitude

$$|\vec{a}| = \sqrt{{a_R}^2 + {a_\theta}^2 + {a_\varphi}^2} \quad \text{with}$$

$$\vec{a}\begin{cases} a_r = \ddot{R} - R\dot{\theta}^2 - R\dot{\varphi}^2 \\ a_\theta = R\ddot{\theta} + 2\dot{R}\dot{\theta} - R\dot{\varphi}^2 sin\theta\ cos\theta \\ a_\varphi = R\ddot{\varphi} sin\theta + 2\dot{R}\ \dot{\varphi} sin\theta + 2R\ \dot{\varphi}\ \dot{\theta}\ cos\theta \end{cases}$$