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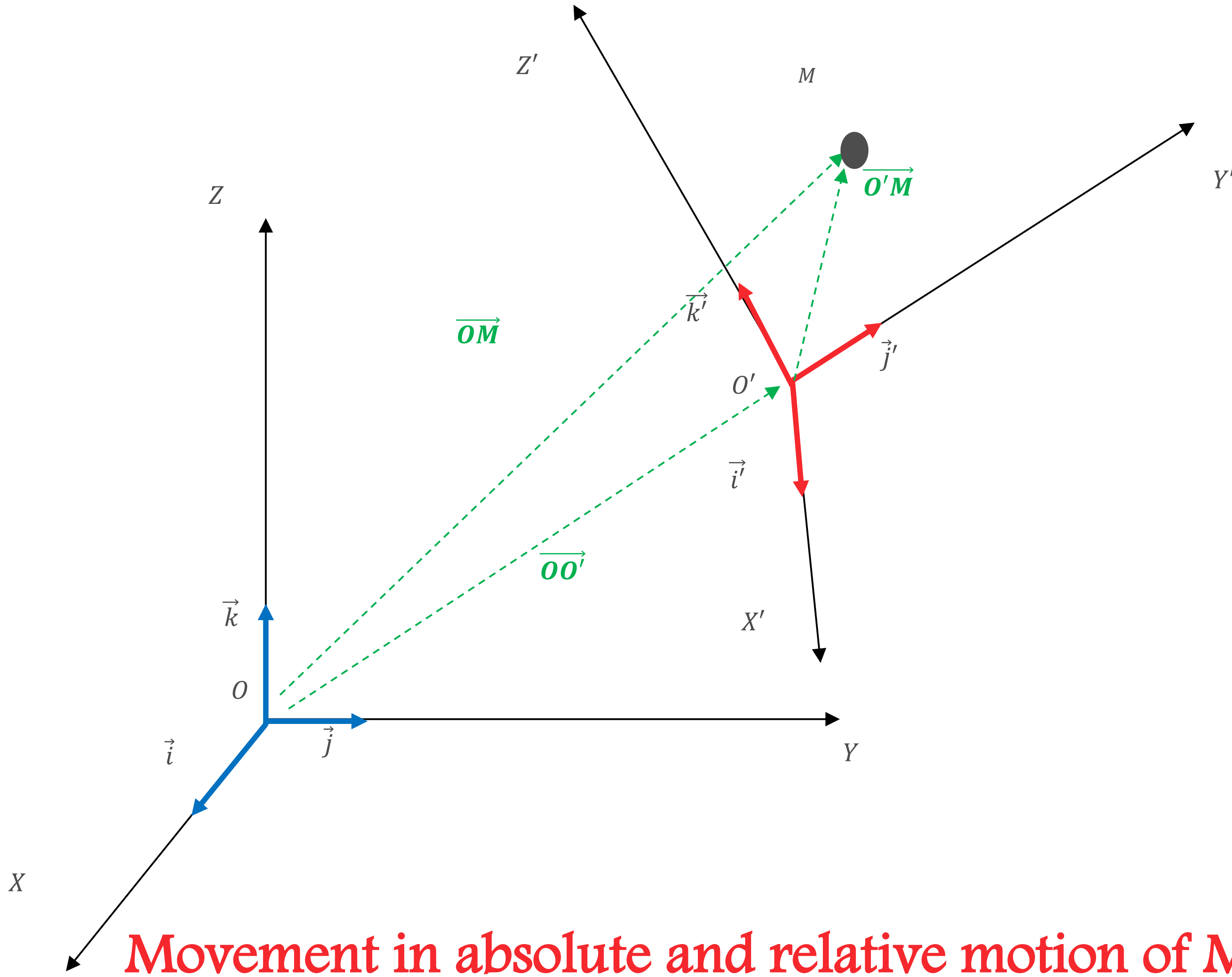
Kinematics with Change of Reference Frame

Relative Motion and Absolute Motion

- We consider two reference frames, $R\ (O\ , X\ , Y\ , Z\ )$ and $R\ '(O\ ', X\ ', Y\ ', Z\ ')$, respective bases $(\vec{i}, \vec{j}, \vec{k})$ and $(\vec{i'}, \vec{j'}, \vec{k'})$, in motion relative to each other. We assume that R is fixed and is called the absolute reference frame. The reference frame $R\ '$ is then called the relative reference frame, as it is in motion with respect to R . We study the motion of a material point M with respect to both reference frames.
- **Example:** A traveler $\mathcal{M}$ on a train, an observer outside the train is considered a R constant (absolute)The train or another within the train is moving with the a moving frame R`



Movement in absolute and relative motion of M

R(o,x,y,z) Absolute Motion of M (Fixe)

R`(o`,x`,y`,z`) Relative Motion of M (MVT)

Absolute Motion of M-point

a) Position Vector:

$$\overrightarrow{O\mathcal{M}} = x\vec{i} + y\vec{j} + z\vec{k}$$

where x,y,z coordinates of the point or ray components of the absolute position at the fixed reference frame (R).

- b) Absolute velocity vector:

- The absolute velocity of M is the velocity of the material point M with respect to the absolute reference frame. It is obtained by taking the derivative with respect to time of the position vector in reference frame R :

$$\vec{v}_a = \frac{d\overrightarrow{O\mathcal{M}}}{dt} / R = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

- The vectors of the base $(\vec{i}, \vec{j}, \vec{k})$, being related to reference frame R have their respective temporal derivatives equal to zero: $\left.\frac{d\vec{i}}{dt}\right|_R = \left.\frac{d\vec{j}}{dt}\right|_R = \left.\frac{d\vec{k}}{dt}\right|_R = \vec{0}$

- Therefore, it is sufficient to differentiate the components:

$$\vec{v}_a = \vec{v} / R = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$$

- c) Absolute acceleration Vector :

- The absolute acceleration is obtained by differentiating the absolute velocity with respect to time in the absolute reference frame:

- $\vec{a}_a = \frac{d\vec{v}_a}{dt} / R = \frac{d^2\overrightarrow{OM}}{dt} / R = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}$
- $\vec{a}_a = \vec{a} \ / R = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}$

- 2.Relative Motion of M:

- Relative position Vector:
- The motion of M with respect to the relative reference frame is called relative motion. The position of point *M* is determined by the Cartesian coordinates in reference frame *R* ’

$$\overrightarrow{O`\mathcal{M}} = x`\overrightarrow{i`} + y`\overrightarrow{j`} + z`\overrightarrow{k}$$

- relative velocity vector:

- The relative velocity of *M* is the velocity of the point mass with respect to the relative reference frame. It is obtained by differentiating the position vector with respect to time in reference frame *R* ’: $\vec{v}_r = \frac{d\overrightarrow{O`\mathcal{M}}}{dt} / R` = \frac{dx`}{dt}\vec{i`} + \frac{dy`}{dt}\vec{j`} + \frac{dz`}{dt}\vec{k`}$

- In this case, the vectors of the base $\left(\vec{i'}, \vec{j'}, \vec{k'}\right)$, being related to reference frame *R* ’ have their respective time derivatives equal to zero: $\left.\frac{d\vec{i'}}{dt}\right|_{R'} = \left.\frac{d\vec{j'}}{dt}\right|_{R'} = \left.\frac{d\vec{k'}}{dt}\right|_{R'} = \vec{0}$

- once again, it is sufficient to differentiate the components:

$$\vec{v}_r = \vec{v} \; / \; R' = \dot{x}'\vec{i'} + \dot{y}'\vec{j'} + \dot{z}'\vec{k'}$$

- c)Relative acceleration Vector:

- The relative acceleration is obtained by differentiating the relative velocity with respect to time in the reference frame *R* ’ :

- $\vec{a}_r = \frac{d\vec{v}_r}{dt} / R` = \frac{d^2\overrightarrow{O`\mathcal{M}}}{dt} / R` = \frac{d^2x`}{dt^2}\overrightarrow{i`} + \frac{d^2y`}{dt^2}\overrightarrow{j`} + \frac{d^2z`}{dt^2}\overrightarrow{k`}$
- $\vec{a}_r = \vec{a} \ / R' = \ddot{x}’\overrightarrow{i'} + \ddot{y}’\overrightarrow{j'} + \ddot{z}’\overrightarrow{k'}$

• Derivation in a moving frame

- $$\overrightarrow{O\mathcal{M}} = x\vec{i} + y\vec{j} + z\vec{k}$$
  $$\overrightarrow{O'\mathcal{M}} = x'\vec{i'} + y'\vec{j'} + z'\vec{k'}$$

By applying a shawl relationship:

- $$\overrightarrow{O\mathcal{M}} = \overrightarrow{OO'} + \overrightarrow{O'\mathcal{M}}$$
- $$\vec{V}_a = \left.\frac{d\overrightarrow{O\mathcal{M}}}{dt}\right|_R = \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \left.\frac{d\overrightarrow{O'\mathcal{M}}}{dt}\right|_R$$
- $$\frac{d\overrightarrow{O'\mathcal{M}}}{dt} = \left.\frac{d\left[x'\vec{i'} + y'\vec{j'} + z'\vec{k'}\right]}{dt}\right|_R$$
- $$\frac{d\overrightarrow{O'\mathcal{M}}}{dt} = \underbrace{\frac{dx'}{dt}\vec{i'} + \frac{dy'}{dt}\vec{j'} + \frac{dz'}{dt}\vec{k'}}_{\vec{V}_r} + x'\left.\frac{d\vec{i'}}{dt}\right|_R + y'\left.\frac{d\vec{j'}}{dt}\right|_R + z'\left.\frac{d\vec{k'}}{dt}\right|_R \vec{V}_r$$

- We have seen that the derivative of a unit vector $\vec{u}$ undergoing rotation with an angular velocity $\vec{\omega}$ with respect to a fixed frame is given by

$\frac{d\vec{u}}{dt} = \vec{\omega} \wedge \vec{u}$. By replacing $\vec{u}$ with the vectors of the base $\left(\vec{i'}, \vec{j'}, \vec{k'}\right)$, we obtain:

• Instantaneous rotation rays

- Instantaneous rotation rays for R` to R (circular motion) is given by:

Any vector rotating with respect to a perpendicular axis its derivative in time is the vector product of its angular velocity $\vec{\omega}$ and rotating vector:

$$\left.\frac{d\vec{i`}}{dt}\right|_{R} = \frac{d\vec{i`}}{d\theta} . \frac{d\theta}{dt} = \vec{\omega} \wedge \vec{i`}$$

- $\frac{d\vec{j`}}{dt} = \vec{\omega} \wedge \vec{j`}$. $\frac{d\vec{k`}}{dt} = \vec{\omega} \wedge \vec{k`}$

Where $\omega = \frac{d\theta}{dt}$ **Secret angle**

- $\frac{d\overrightarrow{O`\mathcal{M}}}{dt} = \underbrace{\dot{x}'\overrightarrow{i'} + \dot{y}'\overrightarrow{j'} + \dot{z}'\overrightarrow{k'}} + x`(\overrightarrow{\omega} \wedge \overrightarrow{i'}) + y'(\overrightarrow{\omega} \wedge \overrightarrow{j'}) + z`(\overrightarrow{\omega} \wedge \overrightarrow{k'})$
- Indeed, we have $\dot{x}'\overrightarrow{i'} + \dot{y}'\overrightarrow{j'} + \dot{z}'\overrightarrow{k'} = \left.\frac{d\overrightarrow{O`\mathcal{M}}}{dt}\right|_{R'}$ which represents the derivative of vector $\overrightarrow{O`\mathcal{M}}$ in the relative reference frame and

$$x`\left(\overrightarrow{\omega} \wedge \overrightarrow{i'}\right) + y'\left(\overrightarrow{\omega} \wedge \overrightarrow{j'}\right) + z`\left(\overrightarrow{\omega} \wedge \overrightarrow{k'}\right) = \overrightarrow{\omega} \wedge \left(x`\overrightarrow{i`} + y`\overrightarrow{j`} + z`\overrightarrow{k`}\right) = \overrightarrow{\omega} \wedge \overrightarrow{O'\mathcal{M}}$$

- This equation shows that the cross product of the angular velocity $\overrightarrow{\omega}$ with the vector $\overrightarrow{O`\mathcal{M}}$ yields the same result as the derivative of $\overrightarrow{O`\mathcal{M}}$ in the reference frame *R* ':
- $\left.\frac{d\overrightarrow{O`\mathcal{M}}}{dt}\right|_{R} = \overrightarrow{\omega}(R'/R) \wedge \overrightarrow{O'\mathcal{M}} + V_r$

- $$\vec{v}_a = \vec{v}_r + \underbrace{\left.\frac{d\overrightarrow{oo`}}{dt}\right|_R + x`\left.\frac{d\vec{i`}}{dt}\right|_R + y`\left.\frac{d\vec{j`}}{dt}\right|_R + z`\left.\frac{d\vec{k`}}{dt}\right|_R}_{\vec{v}_e}$$

$$\vec{v}_a = \vec{v}_r + \vec{v}_e$$

$\vec{v}_a$: s the absolute velocity of the material point

$\vec{v}_r$: is the relative velocity of the material point,

$\vec{v}_e$: is the entrainment velocity.

- accelerations:

- $$\vec{a}_a = \left.\frac{d\vec{v}_a}{dt}\right|_R = \left.\frac{d\vec{v}_r}{dt}\right|_R + \left.\frac{d\vec{v}_e}{dt}\right|_R$$

- $$\vec{a}_a(M) = \left.\frac{d\overrightarrow{v_a}}{dt}\right|_R$$

- $$\left.\frac{d\vec{v}_r}{dt}\right|_R = \frac{d}{dt}\left[\frac{dx`}{dt}\overrightarrow{i`} + \frac{dy`}{dt}\overrightarrow{j`} + \frac{dz`}{dt}\overrightarrow{k`}\right]\Bigg|_R$$

- $$\left.\frac{d\vec{v}_r}{dt}\right|_R = \underbrace{\frac{d^2x`}{dt^2}\overrightarrow{i`} + \frac{d^2y`}{dt^2}\overrightarrow{j`} + \frac{d^2z`}{dt^2}\overrightarrow{k`}}_{\vec{a}_r} + \frac{dx`}{dt}\frac{d\overrightarrow{i`}}{dt} + \frac{dy`}{dt}\frac{d\overrightarrow{j`}}{dt} + \frac{dz`}{dt}\frac{d\overrightarrow{k`}}{dt}$$

- $$\left.\frac{d\vec{v}_r}{dt}\right|_R = \vec{a}_r + \vec{\omega} \wedge \left[\underbrace{\frac{dx`}{dt}\overrightarrow{i`} + \frac{dy`}{dt}\overrightarrow{j`} + \frac{dz`}{dt}\overrightarrow{k`}}_{\vec{v}_r}\right]$$

$$\left.\frac{d\vec{v}_r}{dt}\right|_R = \vec{a}_r + \vec{\omega} \wedge \vec{v}_r$$

- $$\left.\frac{d\vec{v}_e}{dt}\right|_R = \frac{d}{dt}\left[\left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \vec{\omega} \wedge \overrightarrow{O'\mathcal{M}}\right]$$

$$\left.\frac{d\vec{v}_e}{dt}\right|_R = \frac{d^2\overrightarrow{OO'}}{dt^2} + \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O'\mathcal{M}} + \vec{\omega} \wedge \left.\frac{d\overrightarrow{O'\mathcal{M}}}{dt}\right|_R$$

- $$\left.\frac{d\overrightarrow{O'\mathcal{M}}}{dt}\right|_R = \vec{v}_r + \vec{\omega} \wedge \overrightarrow{O'\mathcal{M}}$$
- $$\left.\frac{d\vec{v}_e}{dt}\right|_R = \frac{d^2\overrightarrow{OO'}}{dt^2} + \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O'\mathcal{M}} + \vec{\omega} \wedge (\vec{v}_r + \vec{\omega} \wedge \overrightarrow{O'\mathcal{M}})$$
- $$\left.\frac{d\vec{v}_e}{dt}\right|_R = \frac{d^2\overrightarrow{OO'}}{dt^2} + \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O'\mathcal{M}} + \vec{\omega} \wedge \vec{v}_r + \vec{\omega} \wedge (\vec{\omega} \wedge \overrightarrow{O'\mathcal{M}})$$

- Special case: R’ in rectilinear translation with respect to R
- In this case, the vectors of the relative base $\left(\vec{i}', \vec{j}', \vec{k}'\right)$ are also fixed with respect to the reference frame R :
- $\left.\frac{d\vec{i}'}{dt}\right|_R = \left.\frac{d\vec{j}'}{dt}\right|_R = \left.\frac{d\vec{k}'}{dt}\right|_R = \vec{0}$
- $\vec{v}_e = \left.\frac{d\overrightarrow{oo`}}{dt}\right|_R$
- $\vec{a}_e = \left.\frac{d^2\overrightarrow{oo`}}{dt^2}\right|_R$
- $\vec{a}_e = 0$

$$\vec{a}_a = \vec{a}_r + \vec{\omega} \wedge \vec{v}_r + \frac{d^2\overrightarrow{OO`}}{dt^2} + \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O`\mathcal{M}} + \vec{\omega} \wedge \vec{v}_r + \vec{\omega}$$

$$\vec{a}_a = \vec{a}_r + \underbrace{2\vec{\omega} \wedge \vec{v}_r}_{\vec{a}_c} + \underbrace{\frac{d^2\overrightarrow{OO`}}{dt^2} + \vec{\omega} \wedge \overrightarrow{O`\mathcal{M}} + \vec{\omega}}_{\vec{a}_e}$$

$$\vec{a}_a = \vec{a}_r + \vec{a}_c + \vec{a}_e$$

- $\vec{a}_a = \vec{a}_r + \vec{a}_e + \vec{a}_c$
- $\vec{a}_a$ is the absolute acceleration of the material point
- $\vec{a}_c$: is the complementary acceleration, also known as the Coriolis acceleration. $\vec{a}_c = 2\vec{\omega} \wedge \vec{v}_r$
- $\vec{a}_r$: is the relative acceleration of the material point

$$\vec{a}_r = \frac{d\vec{v}_r}{dt}$$

- $\vec{a}_e$: The entrainment acceleration, denoted as ae , is the acceleration of entrainment.

- $\vec{a}_e = \frac{d^2\overrightarrow{OO`}}{dt^2} + \vec{\dot{\omega}}_{\frac{R}{R`}} \wedge \overrightarrow{O`\mathcal{M}} + \vec{\omega}_{\frac{R}{R`}} \wedge (\vec{\omega}_{\frac{R}{R`}} \wedge \overrightarrow{O`\mathcal{M}})$

- Special case: R' in rectilinear translation with respect to R
- In this case, the vectors of the relative base $\left(\vec{i}', \vec{j}', \vec{k}'\right)$ are als
fixed with respect to the reference frame R :
- $\left.\frac{d\vec{i}'}{dt}\right|_R = \left.\frac{d\vec{j}'}{dt}\right|_R = \left.\frac{d\vec{k}'}{dt}\right|_R = \vec{0}$
- $\vec{v}_e = \left.\frac{d\overrightarrow{oo`}}{dt}\right|_R$
- $\vec{a}_e = \left.\frac{d^2\overrightarrow{oo`}}{dt^2}\right|_R$
- $\vec{a}_e = 0$

- Special case: R' in rotation with respect to R
- If the reference frame *R* ' is rotating with respect to reference frame *R* with an angular velocity $\overrightarrow{\omega}\,(R'/R)$. then the vectors of the relative base are also rotating with the same angular velocity $\overrightarrow{\omega}\,(R'/R) = \omega$.

- $\left.\frac{d\overrightarrow{i'}}{dt}\right|_R = \overrightarrow{\omega} \wedge \overrightarrow{i'}$ $\qquad \left.\frac{d\overrightarrow{j'}}{dt}\right|_R = \overrightarrow{\omega} \wedge \overrightarrow{j'}$ $\qquad \left.\frac{d\overrightarrow{k'}}{dt}\right|_R = \overrightarrow{\omega} \wedge \overrightarrow{k'}$