Course of **Analysis I**
Chapter I: Properties of the set $\mathbb{R}$

Academic Year 2025-2026

Contents

1 Introduction

In mathematics, the real numbers, denoted by $\mathbb{R}$, are all numbers that belong to either the set of rational numbers or the set of irrational numbers. The set $\mathbb{R}$ is a totally ordered field and additionally satisfies the least upper bound property that underlies real analysis.

2 Common Number Sets

We recall the usual notations for number sets:

- $\mathbb{N}$ is the set of natural numbers: $\{0, 1, 2, \dots\}$.
- $\mathbb{Z}$ is the set of integers: $\{\dots, -2, -1, 0, 1, 2, \dots\}$.
- $\mathbb{Q}$ is the set of rational numbers, i.e. $\mathbb{Q} = \left\{\frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{N} \setminus \{0\}\right\}$.
- $\mathbb{R}$ denotes the set of real numbers, with the inclusions $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.
- The set $\mathbb{R} \setminus \mathbb{Q}$ is called the set of irrational numbers.
- For each of these sets, adding the superscript * means we exclude zero from the set: $\mathbb{N}^*$, $\mathbb{Z}^*$, $\mathbb{Q}^*$, and $\mathbb{R}^*$.

3 Intervals in $\mathbb{R}$

Let a, b be two real numbers. We define the following sets, called intervals of $\mathbb{R}$

$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}, \quad [a, b[= \{x \in \mathbb{R} : a \leq x < b\}$

$]a, b] = \{x \in \mathbb{R} : a < x \leq b\}, \quad]a, b[= \{x \in \mathbb{R} : a < x < b\}$

$[a, +\infty[= \{x \in \mathbb{R} : a \leq x\}, \quad]-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$

$]a, +\infty[= \{x \in \mathbb{R} : a < x\}, \quad]-\infty, b[= \{x \in \mathbb{R} : x < b\}$

$\mathbb{R}^*_+ =]0, +\infty[, \quad \mathbb{R}^+ = [0, +\infty[, \quad \mathbb{R}^- =]-\infty, 0], \quad \mathbb{R}^*_- =]-\infty, 0[$

$\mathbb{R} =]-\infty, +\infty[, \quad \mathbb{R}^* =]-\infty, 0[\cup]0, +\infty[$ is not an interval.

4 Upper Bound, Lower Bound, and Bounded Sets

Definition 1. *Let E be a non-empty subset of $\mathbb{R}$*

1) *An element* $M \in \mathbb{R}$ *is called an* upper bound *of the set* E *if and only if*

$$\forall x \in E : x \leq M.$$

In this case, E *is said to be* upper bounded.

2) *An element* $m \in \mathbb{R}$ *is called a* lower bound *of the set* E *if and only if*

$$\forall x \in E : x \geq m.$$

In this case, E *is said to be* lower bounded.

3) *The set* E *is said to be* bounded *if it is both upper and lower bounded; that is,*

$$\exists m, M \in \mathbb{R} : \forall x \in E, \quad m \leq x \leq M.$$

Example 1. *Let*

$$E =]1, 4[, \quad A = [0, +\infty).$$

Determine (if they exist) the upper bounds and lower bounds of E *and* A.
We have:

- *Upper bounds of* $]1, 4[$ *are:* $[4, +\infty[$.
- *Lower bounds of* $]1, 4[$ *are:* $]-\infty, 1]$.
- *There are no upper bounds of* $[0, +\infty[$.
- *Lower bounds of* $[0, +\infty[$ *are* $]-\infty, 0]$.

5 Maximum and Minimum Element

Definition 2. *Let* E *be a non-empty subset of* $\mathbb{R}$

1) *An element* $a \in E$ *is called the* maximum *of* E *if and only if*

$$\forall x \in E : x \leq a.$$

We write $\max E = a$.

2) *An element* $b \in E$ *is called the* minimum *of* E *if and only if*

$$\forall x \in E : b \leq x.$$

We write $\min E = b$.

Example 2. *Consider the sets:* $A = [0, 1), \quad B = \{-3, 2, 5, 8\}$.

- *The maximum element of* A *does not exist.*
- *The minimum element of* A *is:* $\min A = 0$.
- *The maximum element of* B *is:* $\max B = 8$.
- *The minimum element of* B *is:* $\min B = -3$.

6 Supremum and Infimum

Definition 3. *Let E be a non-empty subset of $\mathbb{R}$*

1) *The* supremum *(least upper bound) of E is defined as the minimum of the set of upper bounds of E.*

2) *The* infimum *(greatest lower bound) of E is defined as the maximum of the set of lower bounds of E.*

Theorem 1.3 (Supremum and Infimum Theorem)

1) Every non-empty and bounded above subset of $\mathbb{R}$ admits a supremum.

2) Every non-empty and bounded below subset of $\mathbb{R}$ admits an infimum.

Example 3. *Let $A = [-1, 1]$.*

- *The upper bounds of A are:* $[1, +\infty)$.
- *The lower bounds of A are:* $(-\infty, -1]$.
- *The supremum of A is:* $\sup(A) = 1$.
- *The infimum of A is:* $\inf(A) = -1$.
- *The greatest element of A is:* $\max(A) = 1$.
- *The smallest element of A is:* $\min(A) = -1$.

non-empty subset of $\mathbb{R}$ that is bounded below has an infimum.

Remark 1. 1. *If the maximum $\max A$ (respectively, minimum $\min A$) exists, then $\sup A = \max A$ (respectively, $\inf A = \min A$).*

2. *If the supremum $\sup A$ (respectively, infimum $\inf A$) belongs to A, then $\max A = \sup A$ (respectively, $\min A = \inf A$).*

3. *If the supremum $\sup A$ (respectively, infimum $\inf A$) does not belong to A, then the maximum $\max A$ (respectively, minimum $\min A$) does not exist.*

Note: The supremum of a bounded above set A (respectively, the infimum of a bounded below set A) always exists but may not belong to A. On the other hand, the maximum of a bounded above set (respectively, the minimum of a bounded below set) may not exist.

Example 4. *Let $A =]-4, 6]$, a bounded subset of $\mathbb{R}$: We have the following properties:*

- *The upper bounds of A are:* $[6, +\infty[$.
- *The lower bounds of A are:* $]-\infty, -4]$.
- *The maximum element of A is:* $\max(A) = 6$.

- *The minimum element of A does not exist because $-4 \notin A$.*

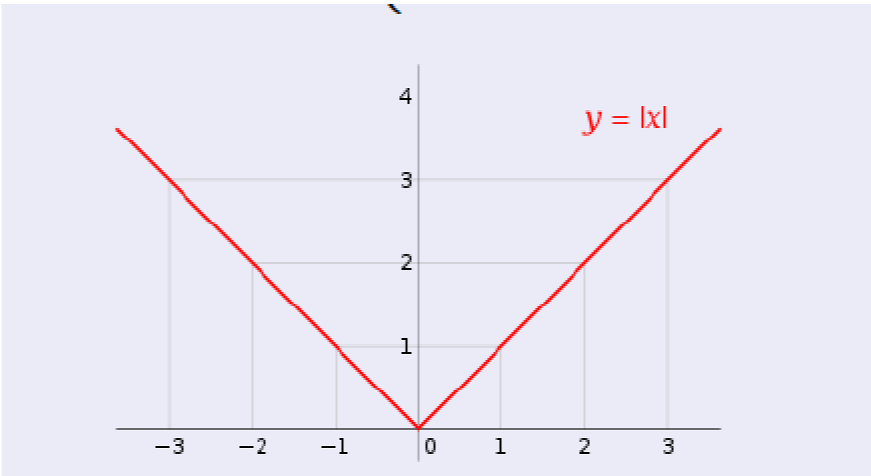
- *The supremum of A is:* $\sup(A) = 6$.

- *The infimum of A is:* $\inf(A) = -4$.

7 Absolute Value

Definition 4. *The absolute value is a mapping from $\mathbb{R}$ to the set of non-negative real numbers $\mathbb{R}^+$, denoted by $|\cdot|$ and defined by:*

$$|\cdot| : \mathbb{R} \to \mathbb{R}^+, \quad x \mapsto |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$



Example 5. *We have:*

$$|7| = 7, \quad |-3| = -(-3) = 3.$$

For any $x \in \mathbb{R}$,

$$|x-1| = \begin{cases} x-1 & \text{if } x \geq 1, \\ -(x-1) = -x+1 & \text{if } x < 1. \end{cases}$$

7.1 Properties of Absolute Value

1. $\forall x \in \mathbb{R} : |x| = \max\{-x, x\}$.

2. $\forall x \in \mathbb{R} : |-x| = |x|$.

3. $\forall x \in \mathbb{R} : (|x| = 0) \iff (x = 0)$.

4. $\forall x, y \in \mathbb{R} : |xy| = |x| \cdot |y|$.

5. $\forall x \in \mathbb{R}, \forall n \in \mathbb{N} : |x^n| = |x|^n$.

6. $\forall x, y \in \mathbb{R} : |x+y| \leq |x| + |y|$ (First triangle inequality).

7. $\forall x, y \in \mathbb{R} : ||x| - |y|| \leq |x-y|$ (Second triangle inequality).

7.2 Usual Distance on $\mathbb{R}$

Definition 5. *The usual distance on $\mathbb{R}$ is the function*

$$d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+, \quad d(x,y) = |x-y|,$$

where $d(x,y)$ is called the distance between x and y.

Properties:

1. $\forall x, y \in \mathbb{R} : \big(d(x,y) = 0 \iff x = y\big)$.

2. $\forall x, y \in \mathbb{R} : d(x,y) = d(y,x)$.

3. $\forall x, y, z \in \mathbb{R} : d(x,z) \leq d(x,y) + d(y,z)$.

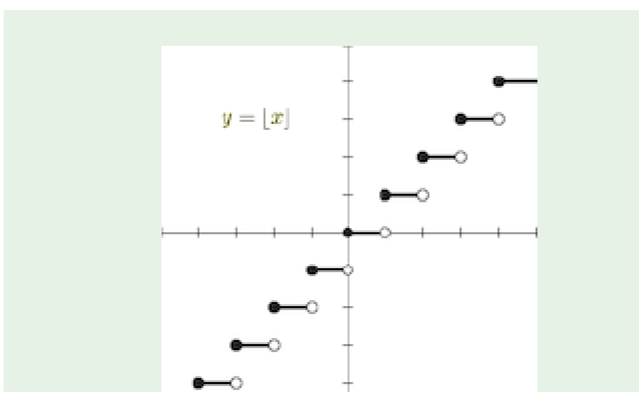
8 Integer Part (Floor Function)

Definition 6. *For every real number $x \in \mathbb{R}$, there exists a unique integer $[x] \in \mathbb{Z}$ such that:*

$$[x] \leq x < [x] + 1.$$

The integer $[x]$ is called the integer part *or* floor *of the real number x, and is denoted also by $E(x)$.*

Example 6. *1)* $\lfloor 3.640869 \rfloor = 3, \quad \lfloor -3.640869 \rfloor = -4.$



2) $[1.5] = 1, \quad [4] = 4, \quad [3.7] = 3,$

$[-1.5] = -2, \quad [1.9] = 1, \quad [-3.7] = -4.$

Properties

Let $x \in \mathbb{R}$. Then we have:

1. $[x] \leq x < [x] + 1$.
2. $x - 1 < [x] \leq x$.
3. $([x] = x) \iff (x \in \mathbb{Z})$.
4. $\forall n \in \mathbb{Z} : \lfloor x + n \rfloor = n + \lfloor x \rfloor$.
5. $\forall x, y \in \mathbb{R}, [x] + [y] \leq [x + y] \leq [x] + [y] + 1$,
6. $x \leq y \implies [x] \leq [y]$.

9 Exercicse

Exercice 1. *Let E be the set defined by*

$$E = \left\{ \frac{1}{x^2 + 1} \mid x \in]0, 1] \right\}.$$

Show that E is bounded.
We have:

$$x \in]0, 1] \implies 0 < x \leq 1 \implies 0 < x^2 \leq 1 \implies 1 < 1 + x^2 \leq 2.$$

Therefore,

$$\frac{1}{2} \leq \frac{1}{1 + x^2} < 1.$$

Thus, for every $a \in E$, we have

$$\frac{1}{2} \leq a < 1,$$

which shows that E is bounded. Moreover, we have:

- *The upper bounds (majorants) of E are: $[1, +\infty[$.*
- *The lower bounds (minorants) of E are: $\left]-\infty, \frac{1}{2}\right]$.*
- *The supremum of E is: $\sup(E) = 1$.*
- *The infimum of E is: $\inf(E) = \frac{1}{2}$.*
- *The greatest element of E does not exist.*
- *The smallest element of E is: $\min E = \frac{1}{2}$.*