TABLE DES MATIÈRES

Introduction générale

CHAPITRE 1

Methods of mathematical reasoning

1.1 Mathematical Logic

1.1.1 Assertions

An assertion is a sentence that is either true or false, not both at the same time.

Exemple 1.1.1 **(a)** $2+2=4$ *is a true assertion.*

(b) *For every* $z \in \mathbb{C}$ *we have* $|z|=1$ *is a false assertion.*

1.1.2 Mathematical logical operators

If P is an assertion and Q is another assertion, we will define new assertions constructed from P and Q.

The logical operator and ($\wedge$)

The assertion ‹‹P et Q›› is true if P is true and Q is true.
The assertion ‹‹P et Q›› is false otherwise.

We summarize this in a truth table :

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

(a) $(3 + 5 = 8) \wedge (3 \times 6 = 18)$ is a true assertion.

(b) $(2 + 2 = 4) \wedge (2 \times 3 = 7)$ is a false assertion.

The logical operator or $(\vee)$

The assertion ‹‹P or Q›› is true if one of the two assertions P or Q is true.
The assertion ‹‹P or Q›› is false if both assertions P and Q are false.
We repeat this in the truth table :

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Exemple 1.1.2 **(a)** $(2 + 2 = 4 \vee 3 \times 2) = 6$ *is a true assertion.*

(b) $(2 = 4 \vee 4 \times 2 = 7)$ *is a false assertion.*

Negation $\overline{\mathbf{P}}$

The assertion $\overline{\mathbf{P}}$ is true if P is false, and false if P is true.

P	$\overline{P}$
T	F
F	T

Exemple 1.1.3 *The negation of the assertion* $3 \geqslant 0$ *she is the assertion* $3 < 0$.

Implication $\Longrightarrow$

The mathematical definition is as follows :
The assertion ($\overline{P}$ Or Q) is noted ‹‹$P \Longrightarrow Q$››
Its truth table is therefore the following :

P	Q	$P \Longrightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Exemple 1.1.4 $2+2=5 \Rightarrow \sqrt{2}=2$ *is true! Yes, if P is false then the assertion $P \Rightarrow Q$ is always true.*

Equivalence $\Longleftrightarrow$

Equivalence is defined by ($P \Longleftrightarrow Q$) is the assertion ($P \Longrightarrow Q$) and ($Q \Longrightarrow P$).
We will say (P is equivalent to Q) or (P if and only if Q).
This assertion is true when P and Q are true or when P and Q are false.
The truth table is :

P	Q	$P \Longleftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Exemple 1.1.5 *For $x, x^{'} \in \mathbb{R}$, Equivalence $x \cdot x^{'} = 0 \Longleftrightarrow x = 0$ or $x^{'} = 0$ is true.*

1.1.3 Quantifiers

The quantifier $\forall$: "for every "

The assertion

$$\forall x \in E,\ P(x)$$

is a true assertion when the assertions $P(x)$ are true for all elements x of the set E.

We read : For all x in E, $P(x)$ is true.

Exemple 1.1.6 **(a)** $\forall\ x \in \mathbb{R}, x^2 \geq 0$ *is a true assertion.*

(b) $\forall\ x \in \mathbb{R}, x^2 \geq 1$ *is a false assertion.*

The quantifier $\exists$: "there exists "

The assertion

$$\exists x \in E, P(x)$$

is a true assertion when we can find at least one element x of E for which $P(x)$ is true.

We read : there exists x in E such that $P(x)$ (be true).

Exemple 1.1.7 **(a)** $\exists\ x \in \mathbb{R}, x^2 \leq 0$ *is true, for example $x = 0$.*

(b) $\exists\ x \in \mathbb{R}, x^2 < 0$ *is false.*

The negation of quantifiers

The negation of $(\forall x \in E, P(x))$ is $(\exists x \in E, \overline{P(x)})$.

The negation of $(\exists x \in E, P(x))$ is $(\forall x \in E, \overline{P(x)})$.

1.2 Reasonings

1.2.1 Direct reasoning

We want to show that the assertion $P \Rightarrow Q$ is true. We assume that P is true and we show that then Q is true.

Exemple 1.2.1 *Let $a, b \in \mathbb{R}$, Show that $a = b \Rightarrow \frac{a+b}{2} = b$.*

Let's take $a = b$, then $\frac{a}{2} = \frac{b}{2}$, so

$$\begin{aligned} \frac{a}{2} + \frac{b}{2} &= \frac{b}{2} + \frac{b}{2} \\ &\Rightarrow \frac{a+b}{2} = b. \end{aligned}$$

1.2.2 Reasoning by contraposition

Reasoning by contraposition is based on the following equivalence :
The assertion $(P \Rightarrow Q)$ is equivalent to $(\overline{Q} \Rightarrow \overline{P})$

$$(P \Rightarrow Q) \Leftrightarrow (\overline{Q} \Rightarrow \overline{P})$$

Exemple 1.2.2 *Let $x \in \mathbb{R}$. Show that*

$$\underbrace{x \neq 2 \text{ and } x \neq -2}_{P} \Rightarrow \underbrace{x^2 \neq 4}_{Q}$$

Demonstration
By contraposition this is equivalent to

$$\underbrace{x^2 = 4}_{\overline{Q}} \Rightarrow \underbrace{x = 2 \text{ or } x = -2}_{\overline{P}}$$

Indeed, let's take $x^2 = 4$, then $(x-2)(x+2) = 0$, so $x = 2$ or $x = -2$.

Exemple 1.2.3 *Let $n \in \mathbb{N}$. Show that if n^2 is even then n is even.*
Demonstration
*By contraposition, we assume that n is not **even**. We want to show that then n^2 is not even.*
*As n is not even, it is **odd** and therefore there exists $k \in \mathbb{N}$ such that $n = 2k+1$.*
Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2k' + 1$, with $k' = 2k^2 + 2k \in \mathbb{N}$. So n^2 is odd.
Conclusion : *we have shown that if n is odd $\Rightarrow n^2$ is odd. By contraposition, this is equivalent to : If n^2 is even $\Rightarrow n$ is even.*

1.2.3 Reasoning by the absurd

The reasoning by the absurd to show $P \Rightarrow Q$, is based on the following principle :
We suppose both that P is true and that Q is false and we search a contradiction.
So if P is true then Q must be true and therefore $P \Rightarrow Q$ is true.

Exemple 1.2.4 *Let $a; b > 0$. Show that if $\frac{a}{1+b} = \frac{b}{1+a} \Rightarrow a = b$.*
Demonstration
We reason with the absurd assuming that $\frac{a}{1+b} = \frac{b}{1+a}$ et $a \neq b$.
This leads to

$$\begin{aligned}\left(\frac{a}{1+b} = \frac{b}{1+a}\right) &\Leftrightarrow a(1+a) = b(1+b)\\ &\Leftrightarrow a^2 - b^2 = -(a-b)\\ &\Leftrightarrow (a-b)(a+b) = -(a-b).\end{aligned}$$

As $a \neq b$ then $a - b \neq 0$ and therefore dividing by $a - b$ we obtain

$$a + b = -1$$

The sum of two positive numbers cannot be negative. We get a contradiction. So we conclude

$$If\ \frac{a}{1+b} = \frac{b}{1+a}\ \text{then } a \neq b.$$

1.2.4 Reasoning by counter example

If we want to show that an assertion of the type $(\forall x \in E, P(x))$ is true then for each x of E we must show that $P(x)$ is true. On the other hand, to show that this assertion is false then it is enough to find $x \in E$ such that $P(x)$ is false.

Exemple 1.2.5 *Show that the following statement is false*

$$\forall x \in \mathbb{R}, x^2 - 1 > 1.$$

A counter example is $x = 0 \in \mathbb{R}$, because $(0)^2 - 1 > 1$ is false.

1.2.5 Reasoning by recurrence

The principle of recurrence allows us to show that an assertion $P(n)$, depending on n, is true for all $n \in \mathbb{N}$.

The recurrence demonstration is done in two steps :

i) We prove $P(0)$ is true.

ii) We assume $n \geq 0$ given with $P(n)$ true, and we then demonstrate that the assertion $P(n+1)$ is true.

Finally, in the conclusion, we recall that by the principle of recurrence $P(n)$ is true for all $n \in \mathbb{N}$.

Exemple 1.2.6 *Show that for all $n \in \mathbb{N} : 2^n > n$.*

Demonstration

Let us note :

$$P(n) : 2^n > n,\ for\ all\ n \in \mathbb{N}.$$

We will demonstrate by recurrence that $P(n)$ is true for all $n \in \mathbb{N}$.

i) *For $n = 0$ we have $2^0 = 1 > 0$, so $P(0)$ is true.*

ii) *Let $n \in \mathbb{N}$, suppose $P(n)$ is true. We will show that $P(n+1)$ is true.*

$$\begin{aligned} 2^{n+1} &= 2^n + 2^n \\ &\geq n + 2^n, \ \textit{because by } P(n) \textit{ we know that } 2^n > n; \\ &\geq n + 1, \ \textit{because } 2^n \geq 1. \end{aligned}$$

So $P(n+1)$ is true.