Course of Maths I
Chapter II: Sets, Relations and applications

Academic Year 2024-2025

Contents

1 Set Theory

Definition 1. *A set is a collection of elements. Among the sets, one particular set is the empty set, denoted by* $\varnothing$.
Let E *be a set, we write* $x \in E$ *if* x *is an element of* E, *and* $x \notin E$ *otherwise.*

Example 1. *We have* $\{0, 1\}$, $\{$*red, black*$\}$, *and* $\{0, 1, 2, \ldots\} = \mathbb{N}$ *are sets. Thus* $0 \in \{0, 1\}$ *and* $2 \notin \{0, 1\}$.

1.1 Inclusion, union, intersection, complement

Definition 2 (Inclusion)**.** *A set* E *is included in a set* F, *if every element of* E *is also an element of* F, *and we write* $E \subset F$. *In other words:*

$$\forall x, x \in E \implies x \in F$$

We then say that E is a subset of F or a part of F.

Example 2. *We have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.*

Definition 3 (Equality)**.** *Two sets E and F are equal if and only if each is included in the other, that is:*

$$E = F \iff E \subset F \text{ and } F \subset E$$

Example 3. *If $E = \mathbb{R}$, we have*

$$A = \{x \in \mathbb{R} : |x-1| \leq 1\} = \{x \in \mathbb{R} : -1 \leq x-1 \leq 1\} = \{x \in \mathbb{R} : 0 \leq x \leq 2\} = [0,2]$$

Definition 4 (The power set of E)**.** *Let E be a set, we form a set called the power set of E, denoted by $\mathcal{P}(E)$, which is characterized by the following relation:*

$$\mathcal{P}(E) = \{A : A \subseteq E\}$$

Example 4. *If $E = \{1,2,3\}$, then*

$$\mathcal{P}(E) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\},$$

so $\{1\} \in \mathcal{P}(E)$ and $E \in \mathcal{P}(E)$.

Definition 5 (Complement)**.** *Let E be a set, the complement of $A \subseteq E$, denoted by $C_E A$ or A^c, is the set of elements of E that do not belong to A, that is:*

$$C_E A = \{x \in E : x \notin A\}$$

Definition 6 (Union and intersection)**.** *The union of two sets A and B, denoted by $A \cup B$, is the set formed by the elements x that belong to A or belong to B, that is:*

$$A \cup B = \{x \in E : x \in A \text{ or } x \in B\}$$

The intersection of two sets A and B, denoted by $A \cap B$, is the set formed by the elements x that belong to A and belong to B, that is:

$$A \cap B = \{x \in E : x \in A \text{ and } x \in B\}$$

Example 5. *If $A = \{1,2,3\}$ and $B = \{2,3,4,5\}$, then $A \cup B = \{1,2,3,4,5\}$ and $A \cap B = \{2,3\}$.*

Definition 7 (Difference and symmetric difference)**.** *Let E be a set, the difference of A and B, denoted by $A \setminus B$, is the set formed by the elements x that belong to A and do not belong to B, that is:*

$$A \setminus B = \{x \in A : x \notin B\}$$

The symmetric difference of A and B, denoted by $A \triangle B$, is the set formed by the elements x that belong to $A \cup B$ and do not belong to $A \cap B$, that is:

$$A \triangle B = (A \cup B) \setminus (A \cap B)$$

Example 6. *If $E = \mathbb{R}$, $A = [0,1]$, and $B =]0, +\infty[$, then*

$$A \setminus B = \{0\}, \quad B \setminus A =]1, +\infty[, \quad \text{and} \quad A \triangle B = \{0\} \cup]1, +\infty[$$

1.2 Cartesian Product

Definition 8. *The Cartesian product of two sets E and F, denoted by $E \times F$, is the set of pairs (x, y) where $x \in E$ and $y \in F$.*

$$E \times F = \{(x, y) : x \in E \text{ and } y \in F\}$$

Example 7. *If $E = \{1, 2\}$ and $F = \{3, 5\}$, then*

$$E \times F = \{(1, 3), (1, 5), (2, 3), (2, 5)\}$$

$$F \times E = \{(3, 1), (3, 2), (5, 1), (5, 2)\} \neq E \times F$$

2 Order Relation, Equivalence Relation

2.1 Binary Relations

Definition 9. *A binary relation on a set E is any assertion between two objects, which can be either verified or not, denoted by $x\mathcal{R}y$, and read as "x is related to y".*

Example 8. *In $\mathbb{R}$, we define the relation R by:*

$$x\mathcal{R}y \quad \Leftrightarrow \quad x - y \geq 0$$

Definition 10. *Let $\mathcal{R}$ be a binary relation on a set E. For all $x, y, z \in E$, we say that $\mathcal{R}$ is:*

1. ***Reflexive**, if every element is related to itself, that is,*

$$x\mathcal{R}x \quad \forall x \in E$$

2. ***Symmetric**, if for all $x, y \in E$, if x is related to y, then y is related to x, that is,*

$$x\mathcal{R}y \Rightarrow y\mathcal{R}x \quad \forall x, y \in E$$

3. ***Transitive**, if for all $x, y, z \in E$, if x is related to y and y is related to z, then x is related to z, that is,*

$$(x\mathcal{R}y \;\wedge\; y\mathcal{R}z) \Rightarrow x\mathcal{R}z \quad \forall x, y, z \in E$$

4. ***Anti-symmetric**, if two elements are related to each other, then they are equal, that is,*

$$(x\mathcal{R}y \;\wedge\; y\mathcal{R}x) \Rightarrow x = y \quad \forall x, y \in E$$

2.2 Equivalence Relation

Definition 11. *A binary relation $\mathcal{R}$ on E is an equivalence relation if it is both reflexive, symmetric, and transitive.*

Definition 12. *Let $\mathcal{R}$ be an equivalence relation on E. The equivalence class of $x \in E$ is defined as the set of elements in E that are related to x by $\mathcal{R}$, denoted by $\overline{x}$ or $cl(x)$ or $\mathcal{C}(x)$:*

$$\mathcal{C}(x) = \{y \in E : y\mathcal{R}x\}.$$

The equivalence class $\mathcal{C}(x)$ is non-empty because $\mathcal{R}$ is reflexive and thus contains at least x. We denote by

$$E/\mathcal{R} = \{\mathcal{C}(x) : x \in E\}$$

the set of equivalence classes of E under the relation $\mathcal{R}$.

Example 9. *In $\mathbb{R}$, we define the relation $\mathcal{R}$ by:*

$$x\mathcal{R}y \iff x - y \in \mathbb{Z}.$$

This relation is indeed an equivalence relation. Indeed,

- *For $x \in \mathbb{R}$: $x\mathcal{R}x \iff 0 \in \mathbb{Z}$, and since $0 \in \mathbb{Z}$, then $x\mathcal{R}x$; $\forall x \in \mathbb{R}$, so R is a reflexive relation.*
- *For $x, y \in \mathbb{R}$, we have*

  $$(x\mathcal{R}y) \iff (x - y \in \mathbb{Z}) \iff (y - x \in \mathbb{Z}) \implies y\mathcal{R}x,$$

  thus $\mathcal{R}$ is a symmetric relation.
- *For $x, y, z \in \mathbb{R}$, we have*

  $$\begin{aligned}(x\mathcal{R}y \ \wedge \ y\mathcal{R}z) &\implies (x - y \in \mathbb{Z} \ \wedge \ y - z \in \mathbb{Z}) \\ &\implies (x - y + y - z \in \mathbb{Z}) \\ &\implies (x - z \in \mathbb{Z}) \implies (x\mathcal{R}z),\end{aligned}$$

  thus $\mathcal{R}$ is a transitive relation.

Thus, the set of equivalence classes $\mathcal{C}(x)$ is given by

$$\begin{aligned}\mathcal{C}(x) &= \{y \in \mathbb{R} : y - x \in \mathbb{Z}\} \\ &= \{y \in \mathbb{R} : y \in x + \mathbb{Z}\} \\ &= \{y \in \mathbb{R} : y = k + x \text{ for } k \in \mathbb{Z}\} \\ &= \{k + x : k \in \mathbb{Z}\}.\end{aligned}$$

If $x \in \mathbb{Z}$, then we have $\mathcal{C}(x) = \mathbb{Z}$.

2.3 Order Relation

Definition 13. *A binary relation $\mathcal{R}$ on E is called an order relation if it is antisymmetric, transitive, and reflexive.*

3 Applications

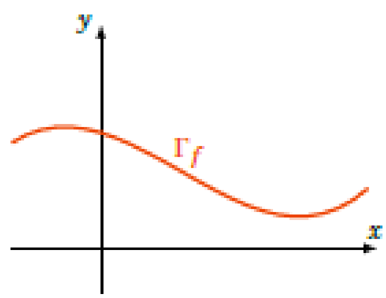
3.1 Definition of an application

Definition 14. *Let E and F be given sets. An application from E to F is any correspondence f between the elements of E and those of F that associates to each element of E exactly one element of F. We write:*

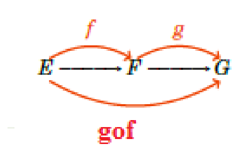
$$f : E \to F$$
$$x \mapsto f(x)$$

• **Equality**: two applications $f, g : E \longrightarrow F$ are equal if and only if $\forall x \in E, f(x) = g(x)$. We then note $f = g$.
• **Graph of** f:$E \longrightarrow F$ is

$$\Gamma_f = \{(x, f(x)) \in E \times F \mid x \in E\}.$$



• **Composition**: Let $f : E \longrightarrow F$ and $g : F \longrightarrow G$, then $g \circ f : E \longrightarrow G$ is the application defined by $g \circ f(x) = g(f(x))$.



• **Identity**: Let E be a set. We call the identity application, denoted $\mathrm{Id} : E \to E$, the function that satisfies

$$\mathrm{Id}(x) = x, \quad \forall x \in E.$$

Example 10. *Let $f : \mathbb{R} \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to [1, +\infty[$ defined by:*

$$f(x) = x^2 \quad \forall x \in \mathbb{R}^+$$

and

$$g(x) = 2x + 1 \quad \forall x \in \mathbb{R}^+.$$

Then the composition $g \circ f : \mathbb{R} \to [1, +\infty[$ *is given by*

$$(g \circ f)(x) = g(f(x)) = g(x^2) = 2x^2 + 1 \quad \forall x \in \mathbb{R}.$$

Example 11. *f and g are two applications defined by:*

$$f : \ ]0, +\infty[\ \to \ ]0, +\infty[, \quad x \mapsto \frac{1}{x} \qquad\qquad g : \ ]0, +\infty[\ \to \mathbb{R}, \quad x \mapsto \frac{x-1}{x+1}$$

Then $g \circ f :]0, +\infty[\longrightarrow \mathbb{R}$ *check for all* $x \in]0, +\infty[$:

$$g \circ f(x) = g(f(x)) = g(\tfrac{1}{x}) = \frac{\frac{1}{x}-1}{\frac{1}{x}+1} = \frac{1-x}{1+x} = -g(x)$$

3.2 Restriction and extension of an application

Definition 15. *Let $A \subset E$ and $f : E \to F$ be an application. The restriction of f to A, denoted $f|_A : E \to F$, is defined by*

$$f|_A(x) = f(x), \quad \text{for all } x \in A.$$

Definition 16. *Let $E \subset G$ and $f : E \to F$ be an application. The extension of f to G is any function g from G to F whose restriction to E is f.*

Example 12. *Given the application*

$$f : \mathbb{R}_+^* \to \mathbb{R}, \quad x \mapsto \ln x,$$

we have

$$g : \mathbb{R}^* \to \mathbb{R}, \quad x \mapsto \ln |x|,$$

and

$$h : \mathbb{R}^* \to \mathbb{R}, \quad x \mapsto \ln(2|x| - x).$$

These are two different extensions of f to $\mathbb{R}$.

3.3 Direct Image, Inverse Image

Let E, F be two sets.

Definition 17. *Let $A \subset E$ and $f : E \to F$. The direct image of A by f is the set*

$$f(A) = \{f(x) : x \in A\} \subset F.$$

Example 13. *Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$, $\forall x \in \mathbb{R}$. If $A = [0, 1]$, then*

$$f([0, 1]) = \{f(x) : x \in [0, 1]\} = \{2x + 1 : x \in [0, 1]\}.$$

We have

$$x \in [0, 1] \implies 0 \le x \le 1 \implies 1 \le 2x + 1 \le 3,$$

thus $f([0, 1]) = [1, 3]$.

Definition 18. *Let $B \subset F$ and $f : E \to F$. The inverse image of B by f is the set*

$$f^{-1}(B) = \{x \in E : f(x) \in B\} \subset E.$$

Example 14. *Let f be the function defined by $f(x) = x^2$ from $\mathbb{R} \to \mathbb{R}^+$. Then*

$$f^{-1}([0,1]) = \{x \in \mathbb{R} : 0 \leq x^2 \leq 1\} = \{x \in \mathbb{R} : 0 \leq |x| \leq 1\} = [-1,1].$$

Let g be defined by $g(x) = \sin(\pi x)$ from $\mathbb{R} \to \mathbb{R}$. Then

$$g^{-1}(\{0\}) = \{x \in \mathbb{R} : \sin(\pi x) = 0\} = \{x : x = k,\ k \in \mathbb{Z}\} = \mathbb{Z}.$$

Proposition 1. *Let E and F be two arbitrary sets and let $f : E \to F$ be an application. For all $A, B \subset E$, the following properties hold:*

1. *$f(A \cap B) \subset f(A) \cap f(B)$.*
2. *$f(A \cup B) = f(A) \cup f(B)$.*

3.4 Injective, surjective, bijective application

Let E, F two sets and $f : E \longrightarrow F$ an application.

Definition 19. *An application f is injective if for all $x_1, x_2 \in E$, whenever $f(x_1) = f(x_2)$, then $x_1 = x_2$. In other words:*

$$\forall x_1, x_2 \in E, \quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

Definition 20. *An application f is surjective if for all $y \in F$, there exists an $x \in E$ such that $y = f(x)$. In other words:*

$$\forall y \in F, \exists x \in E \quad (y = f(x))$$

Example 15. 1. *We consider the application $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by:*

$$\forall x \in \mathbb{R} : \quad f(x) = 2x + 1.$$

Is f injective?
• *Let $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$.*
$f(x_1) = f(x_2) \Longrightarrow 2x_1 + 1 = 2x_2 + 1 \Longrightarrow 2x_1 = 2x_2 \Longrightarrow x_1 = x_2$,
then f is injective.

2. *Using the same example. Is f surjective?.*
Let $y \in \mathbb{R}$, let's try to solve the equation $y = f(x)$.
$y = f(x) \Longleftrightarrow y = 2x + 1 \Longleftrightarrow y - 1 = 2x \Longleftrightarrow x = \dfrac{y-1}{2}$.
Is clear that the expression $\dfrac{y-1}{2}$ is defined for every real y, then f is surjective.

Definition 21. *An application f is bijective if it is both injective and surjective. This is equivalent to: for all $y \in F$, there exists a unique $x \in E$ such that $y = f(x)$. In other words:*

$$\forall y \in F, \quad \exists!\ x \in E \quad y = f(x)$$

Proposition 2. *Let E and F be sets and let $f : E \to F$ be an application.*

1. *The application f is bijective if and only if there exists an application $g : F \to E$ such that*

$$f \circ g = Id_F \qquad \text{and} \qquad g \circ f = Id_E.$$

2. *If f is bijective, then the application g is unique and is also bijective. The application g is called the inverse bijection (or the inverse application) of f and is denoted by f^{-1}. Moreover, we have $(f^{-1})^{-1} = f$.*

Remark 1.

- *$f \circ g = Id_F$ can be reformulated as*

$$\forall y \in F, \quad f(g(y)) = y.$$

- *While $g \circ f = Id_E$ can be expressed as:*

$$\forall x \in E, \quad g(f(x)) = x.$$

- *For example, the application $f : \mathbb{R} \to]0, +\infty[$ defined by $f(x) = \exp(x)$ is bijective, and its inverse bijection is $g :]0, +\infty[\to \mathbb{R}$ defined by $g(y) = \ln(y)$. We have*

$$\exp(\ln(y)) = y, \quad \forall y \in]0, +\infty[\quad \text{and} \quad \ln(\exp(x)) = x, \quad \forall x \in \mathbb{R}.$$

Proposition 3. *Let $f : E \to F$ and $g : F \to G$ be bijective applications. The composition $g \circ f$ is bijective, and its inverse bijection is given by*

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$