# Course of Maths1
# Chapter III: Real functions of a real variable

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## Contents

# 1 Concepts of function

## 1.1 Definitions

**Definition 1.** *A real-valued function of a real variable is a mapping $f : U \longrightarrow \mathbb{R}$ where $U$ is a subset of $\mathbb{R}$.*
*In general, $U$ is an interval or a union of intervals. $U$ is called the domain of definition of the function $f$.*

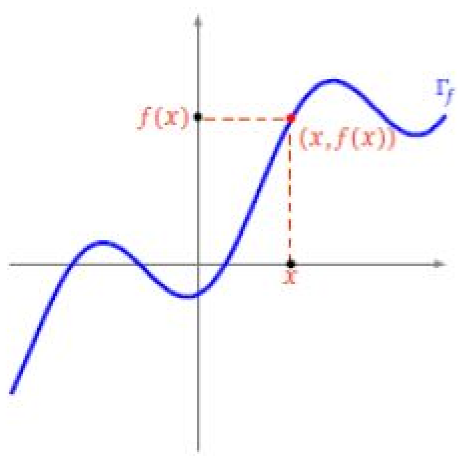
**Example 1.** *The inverse function:*

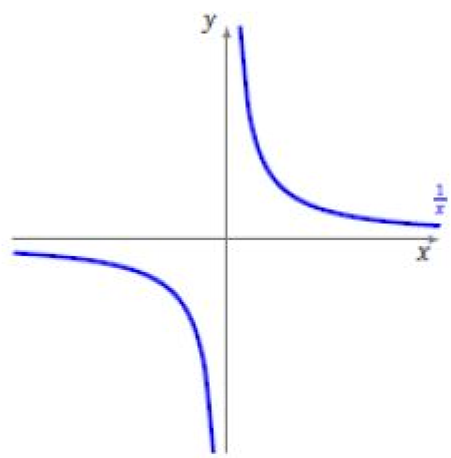
$$\begin{aligned} f :]-\infty, 0[\cup]0, +\infty[ &\longrightarrow \mathbb{R} \\ x &\longmapsto \frac{1}{x} \end{aligned}$$

The graph of a function $f : U \longrightarrow \mathbb{R}$ is the subset $\Gamma_f$ of $\mathbb{R}^2$ defined by

$$\boxed{\Gamma_f = \{(x, f(x)) \mid x \in U\}}$$

The graph of a function (on the left), the example of the graph of $x \longmapsto \frac{1}{x}$ (on the right).



## 1.2 Operations on functions

Let $f : U \longrightarrow \mathbb{R}$ and $g : U \longrightarrow \mathbb{R}$ be two functions defined on the same subset $U$ of $\mathbb{R}$. We can then define the following functions:

- The sum of $f$ and $g$ is the function $f + g : U \longrightarrow \mathbb{R}$ defined by $(f + g)(x) = f(x) + g(x)$ for all $x \in U$.
- The product of $f$ and $g$ is the function $f \times g : U \longrightarrow \mathbb{R}$ defined by $(f \times g)(x) = f(x) \times g(x)$ for all $x \in U$.
- The multiplication by a scalar $\lambda \in \mathbb{R}$ of f is the function $\lambda \cdot f : U \longrightarrow \mathbb{R}$ defined by
  $(\lambda \cdot f)(x) = \lambda \cdot f(x)$ for all $x \in U$.

## 1.3 Bounded functions

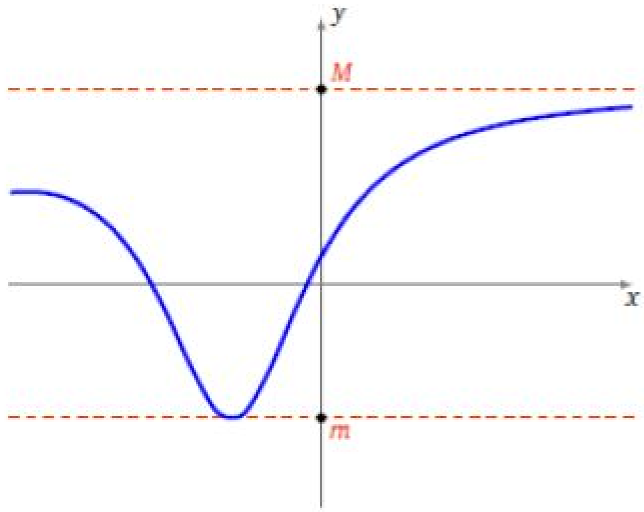
**Definition 2.** *Let $f : U \longrightarrow \mathbb{R}$ and $g : U \longrightarrow \mathbb{R}$ two functions, then:*

- $f \geq g$ *if* $\forall x \in U;\quad f(x) \geq g(x)$*;*
- $f \geq 0$ *if* $\forall x \in U;\quad f(x) \geq 0$*;*
- $f > 0$ *if* $\forall x \in U;\quad f(x) > 0$*;*
- $f$ *is said to be constant on $U$ if* $\exists a \in \mathbb{R},\ \forall x \in U,\ f(x) = a$*;*
- $f$ *is said to be zero on $U$ if* $\forall x \in U,\ f(x) = 0$.

**Definition 3.** *Let $f : U \longrightarrow \mathbb{R}$ be a function. We say that:*

- $f$ *is bounded from above on $U$ if* $\exists M \in \mathbb{R},\ \forall x \in U,\ f(x) \leq M$*;*
- $f$ *is bounded from below on $U$ if* $\exists m \in \mathbb{R},\ \forall x \in U,\ f(x) \geq m$*;*
- $f$ *is bounded on $U$ if it is both upper bounded and lower bounded on $U$, that is, if*
  $\exists M \in \mathbb{R},\ \forall x \in U,\ | f(x) | \leq M$.

Here is the graph of a bounded function (bounded from below by $m$ and from above by $M$).

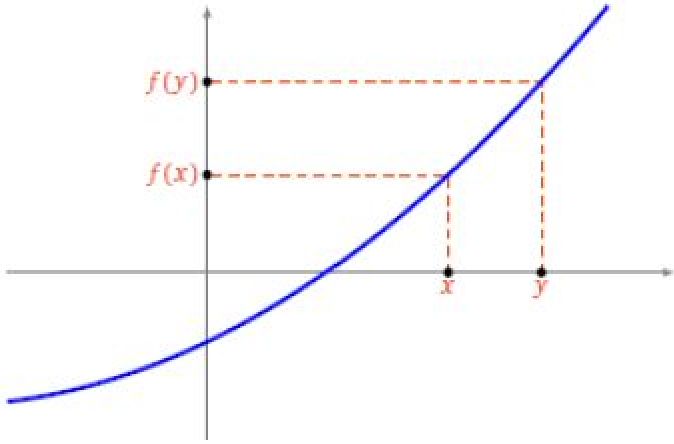



## 1.4 Increasing and decreasing functions

**Definition 4.** *Let $f : U \longrightarrow \mathbb{R}$ be a function. We say that:*

- *$f$ is increasing on $U$ if $\forall\, a, b \in U,\ a \leq b \Rightarrow f(a) \leq f(b)$;*
- *$f$ is strictly increasing on $U$ if $\forall\, a, b \in U,\ a < b \Rightarrow f(a) < f(b)$;*
- *$f$ is decreasing on $U$ if $\forall\, a, b \in U,\ a \leq b \Rightarrow f(a) \geq f(b)$;*
- *$f$ is strictly decreasing on $U$ if $\forall\, a, b \in U,\ a < b \Rightarrow f(a) > f(b)$;*
- *$f$ is monotone (resp. strictly monotone) on $U$ if $f$ is increasing or decreasing (resp. strictly increasing or strictly decreasing) on $U$.*

Here is the graph of a strictly increasing function.




**Example 2.** • *The square root function* $\begin{cases} [0,\infty[ \longrightarrow \mathbb{R} \\ \quad x \longmapsto \sqrt{x} \end{cases}$ *is strictly increasing.*

- *The absolute value function* $\begin{cases} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto \mid x \mid \end{cases}$ *is neither increasing nor decreasing.*

  *However, the function* $\begin{cases} [0,\infty[ \longrightarrow \mathbb{R} \\ \quad x \longmapsto \mid x \mid \end{cases}$ *is strictly increasing.*

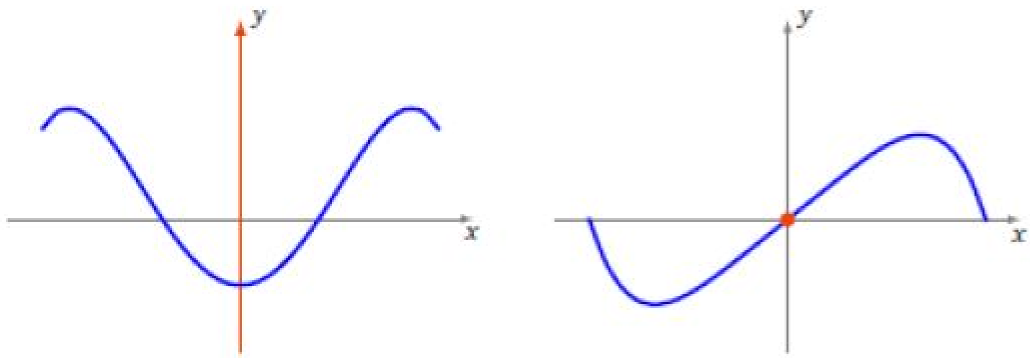
## 1.5 Parity and periodicity

**Definition 5.** *Let $I$ be an interval on $\mathbb{R}$ symmetric with respect to $0$ (that is, of the form $]-a,a[$ or $[-a,a]$ or $\mathbb{R}$ ). Let $f: I \longrightarrow \mathbb{R}$ be a function defined on this interval. We say that:*

- *$f$ is even if $\forall x \in I, \quad f(-x) = f(x)$.*
- *$f$ is odd if $\forall x \in I, \quad f(-x) = -f(x)$.*

**Graphical interpretation:**

- $f$ is even if and only if its graph is symmetric with respect to the y-axis (left figure).
- $f$ is odd if and only if its graph is symmetric with respect to the origin (right figure).




**Example 3.** *The function* $\cos : \mathbb{R} \longrightarrow \mathbb{R}$ *is even. The function* $\sin : \mathbb{R} \longrightarrow \mathbb{R}$ *is odd.*

**Definition 6.** *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be a function and* $T$ *a real number,* $T > 0$. *The function* $f$ *is said to be periodic with period* $T$ *if* $\forall x \in \mathbb{R},\ f(x+T) = f(x)$.

**Example 4.** *The sine and cosine functions are* $2\pi$*-periodic. The tangent function is* $\pi$*-periodic.*

# 2 Limits

## 2.1 Limit at a point

Let $f : I \longrightarrow \mathbb{R}$ be a function defined on an interval $I$ of $\mathbb{R}$. Let $x_0 \in \mathbb{R}$ be a point in $I$ or an endpoint of $I$.

**Definition 7.** *Let* $\ell \in \mathbb{R}$. *It is said that f has a limit* $\ell$ *at* $x_0$ *if*

$$\boxed{\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in I \quad | x - x_0 | < \delta \Longrightarrow | f(x) - \ell | < \varepsilon}$$

*It is also said that* $f(x)$ *tends to* $\ell$ *as* $x$ *approaches* $x_0$. *We denote this as* $\lim\limits_{x \to x_0} f(x) = \ell$ *or* $\lim\limits_{x_0} f = \ell$

**Definition 8.**

- *It is said that f has a limit of* $+\infty$ *at* $x_0$ *if*

$$\forall A > 0,\ \exists \delta > 0,\ \forall x \in I \quad | x - x_0 | < \delta \Longrightarrow f(x) > A.$$

*We then denote* $\lim\limits_{x \to x_0} f(x) = +\infty$.

- *It is said that f has a limit of $-\infty$ at $x_0$ if*

$$\forall A > 0,\ \exists \delta > 0,\ \forall x \in I \quad | x - x_0 |< \delta \Longrightarrow f(x) < -A.$$

*We then denote* $\lim\limits_{x \to x_0} f(x) = -\infty$.

## 2.2 Limit at infinity

Let $f : I \longrightarrow \mathbb{R}$ be a function defined on an interval of the form $I = ]a, +\infty[$.

**Definition 9.**

- *Let $\ell \in \mathbb{R}$. It is said that f has a limit $\ell$ at $+\infty$ if*

$$\forall \varepsilon > 0,\ \exists B > 0,\ \forall x \in I \quad x > B \Longrightarrow | f(x) - \ell |< \varepsilon$$

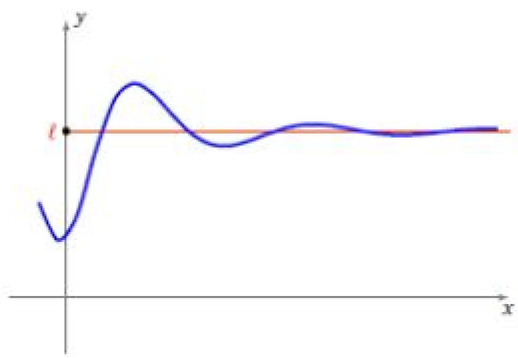
*We then denote* $\lim\limits_{x \to +\infty} f(x) = \ell$ *or* $\lim\limits_{+\infty} f = \ell$.

- *It is said that f has a limit $+\infty$ at $+\infty$ if*

$$\forall A > 0,\ \exists B > 0,\ \forall x \in I \quad x > B \Longrightarrow f(x) > A$$

*We then denote* $\lim\limits_{x \to +\infty} f(x) = +\infty$.

*In the same way, we would define the limit at $-\infty$ for functions defined on intervals of the type $] - \infty, a[$.*




**Example 5.** *We have the following classical limits for all $n \geq 1$:*

- $\lim\limits_{x \to +\infty} x^n = +\infty,$ *and* $\lim\limits_{x \to -\infty} x^n = \begin{cases} +\infty \text{ if } n \text{ is even} \\ -\infty \text{ if } n \text{ is odd} \end{cases}$
- $\lim\limits_{x \to +\infty} \frac{1}{x^n} = 0,$ *and* $\lim\limits_{x \to -\infty} \frac{1}{x^n} = 0.$

### 2.2.1 Right-hand and Left-hand Limits

Let $f$ be a function defined on a set of the form $]a, x_0[ \cup ]x_0, b[$.

**Definition 10.**
- *We call the right-hand limit at $x_0$ of $f$ the limit of the function $f\Big|_{]x_0,b[}$ at $x_0$, and it is denoted as $\lim\limits_{x_0^+} f$.*
- *We similarly define the left-hand limit at $x_0$ of $f$ as the limit of the function $f\Big|_{]a,x_0[}$ at $x_0$, and it is denoted as $\lim\limits_{x_0^-} f$.*
- *We also denote $\lim\limits_{\substack{x \longrightarrow x_0 \\ >}} f(x)$ for the right-hand limit and $\lim\limits_{\substack{x \longrightarrow x_0 \\ <}} f(x)$ for the left-hand limit.*

# 3 Uniqueness of the limit

**Proposition 1.** $\boxed{\textit{If a function has a limit, then that limit is unique.}}$

**Proposition 2.**

$$\boxed{\lim_{x \longrightarrow x_0} f(x) = \ell \iff \lim_{\substack{x \longrightarrow x_0 \\ <}} f(x) = \lim_{\substack{x \longrightarrow x_0 \\ >}} f(x) = \ell}$$

**Example 6.**

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} 2x+3 & if \quad x \geq 0 \\ 4x+5 & if \quad x < 0 \end{cases} \end{aligned}$$

*We have*
$\lim\limits_{\substack{x \longrightarrow 0 \\ >}} f(x) = 3$ *and* $\lim\limits_{\substack{x \longrightarrow 0 \\ <}} f(x) = 5$. *In this case, it is said that $f$ does not have a limit at 0.*

Let there be two functions $f$ and $g$. We assume that $x_0$ is a real number, or that $x_0 = \pm\infty$

**Proposition 3.** *If* $\lim\limits_{x_0} f = \ell \in \mathbb{R}$ *and* $\lim\limits_{x_0} g = \ell' \in \mathbb{R}$, *then:*

- $\lim\limits_{x_0} (\lambda \cdot f) = \lambda \cdot \ell$ *for all* $\lambda \in \mathbb{R}$
- $\lim\limits_{x_0} (f + g) = \ell + \ell'$
- $\lim\limits_{x_0} (f \times g) = \ell \times \ell'$
- *If* $\ell \neq 0$, *then* $\lim\limits_{x_0} \frac{1}{f} = \frac{1}{\ell}$
  *Moreover, if* $\lim\limits_{x_0} f = +\infty$ *(or* $-\infty$ *), then* $\lim\limits_{x_0} \frac{1}{f} = 0$.
- *If* $h$ *is a bounded function and* $\lim\limits_{x_0} f = 0$, *then* $\lim\limits_{x_0} (h \cdot f) = 0$

**Proposition 4** (Composition of Limits)**.**

$$\boxed{\text{If } \lim_{x_0} f = \ell \text{ and } \lim_{\ell} g = \ell', \quad \text{then } \lim_{x_0} g \circ f = \ell'}$$

**Proposition 5.**

- *If* $f \leq g$ *and if* $\lim\limits_{x_0} f = \ell \in \mathbb{R}$ *and* $\lim\limits_{x_0} g = \ell' \in \mathbb{R}$, *then* $\ell \leq \ell'$.
- *If* $f \leq g$ *and if* $\lim\limits_{x_0} f = +\infty$, *then* $\lim\limits_{x_0} g = +\infty$.
- *Sandwich Theorem (Squeeze Theorem)*

$$\boxed{\text{If } f \leq g \leq h \text{ and } \lim_{x_0} f = \lim_{x_0} h = \ell \in \mathbb{R}, \text{ then } g \text{ has a limit at } x_0 \text{ and } \lim_{x_0} g = \ell.}$$

**Proposition 6.**

1. $\lim\limits_{x \to 0} (1 + x)^{\frac{1}{x}} = (1)^{\infty} \stackrel{I.F}{=} e$
2. $\lim\limits_{x \to +\infty} \left(1 + \frac{1}{x}\right)^{x} = (1)^{\infty} \stackrel{I.F}{=} e$

*Proof.*

1. $(1 + x)^{\frac{1}{x}} = \exp\left(\frac{1}{x} \ln (1 + x)\right)$
   and we have :
   $$\lim_{x \to 0} \frac{\ln (1 + x)}{x} = \lim_{x \to 0} \frac{\ln (1 + x) - ln(1)}{x - 0} = (\ln (1 + x))'_{x=0} = \left(\frac{1}{1 + x}\right)_{x=0} = 1,$$
   then, $\lim\limits_{x \to 0} (1 + x)^{\frac{1}{x}} = \exp(1) = e$.
2. The second is proven in the same way.

$\square$

**Example 7.** *1.* $\lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{x}}$.

*2.* $\lim\limits_{x\to+\infty}\left(\frac{x+4}{x-6}\right)^{x}$.

**Solution:**

1.

$$(1+\sin x)^{\frac{1}{x}} = (1+\sin x)^{\frac{1}{x}\cdot\frac{\sin x}{\sin x}} = \left(\underbrace{(1+\sin x)^{\frac{1}{\sin x}}}_{e}\right)^{\frac{\sin x}{x}} \underset{x\to 0}{\longrightarrow} e^{\frac{\sin x}{x}} = e$$

2.

$$\left(\frac{x+4}{x-6}\right)^{x} = \left(\frac{x-6+6+4}{x-6}\right)^{x} = \left(1+\frac{1}{\frac{x-6}{10}}\right)^{x\cdot\frac{x-6}{10}\cdot\frac{10}{x-6}}$$
$$= \left(\left(1+\frac{1}{\frac{x-6}{10}}\right)^{\frac{x-6}{10}}\right)^{\frac{10x}{x-6}} \longrightarrow e^{\frac{10x}{x-6}} = e^{10}$$

# 4 Continuity at a point

## 4.1 Definitions

Let $I$ be an interval in $\mathbb{R}$, and $f : I \longrightarrow \mathbb{R}$ be a function.

**Definition 11.** • *It is said that $f$ is continuous at a point $x_0 \in I$ if*

$$\boxed{\forall\varepsilon>0,\ \exists\delta>0,\ \forall x\in I \quad |\, x-x_0 \,|<\delta \Longrightarrow |\, f(x)-f(x_0)\,|<\varepsilon}.$$

*That is,*

$$\boxed{\lim_{x\longrightarrow x_0} f(x)=f(x_0)}$$

- *We say that $f$ is continuous on $I$ if $f$ is continuous at every point in $I$.*

**Definition 12.** 

- *We say that $f$ is right-continuous at point $x_0 \in I$ if*

$$\boxed{\lim_{\substack{x \longrightarrow x_0 \\ >}} f(x) = f(x_0)}$$

- *We say that $f$ is left-continuous at point $x_0 \in I$ if*

$$\boxed{\lim_{\substack{x \longrightarrow x_0 \\ <}} f(x) = f(x_0)}$$

**Example 8.**

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x \quad &\longmapsto \begin{cases} 2x+1 & if \quad x > 1 \\ 3 & if \quad x = 1 \\ 4x+5 & if \quad x < 1 \end{cases} \end{aligned}$$

*We have*
$\lim_{\substack{x \longrightarrow 1 \\ >}} f(x) = 3 = f(1)$ *and* $\lim_{\substack{x \longrightarrow 1 \\ <}} f(x) = 9 \neq f(1)$. *In this case, it is said that $f$ does not have a limit at 1.*
*$f$ is right-continuous at 1 but is not left-continuous at 1, so $f$ is not continuous at 1.*

**Example 9.** *The following functions are continuous:*

- *The functions "square root" $x \mapsto \sqrt{x}$ and $\ln$ are continuous on $]0, +\infty[$.*
- *The functions $\sin$, $\cos$, $\exp$ and absolute value $x \mapsto \mid x \mid$ are continuous on $\mathbb{R}$.*

**Proposition 7.** *Let $f, g : I \longrightarrow \mathbb{R}$ be two functions continuous at a point $x_0 \in I$. Then*

- *$\lambda \cdot f$ is continuous at $x_0$ (for all $\lambda \in \mathbb{R}$),*
- *$f + g$ is continuous at $x_0$,*
- *$f \times g$ is continuous at $x_0$,*
- *If $f(x_0) \neq 0$, then $\frac{1}{f}$ is continuous at $x_0$.*

**Proposition 8.** *Let $f : I \longrightarrow \mathbb{R}$ and $g : J \longrightarrow \mathbb{R}$ be two functions such that $f(I) \subset J$. If $f$ is continuous at a point $x_0 \in I$, and $g$ is continuous at $f(x_0)$, then $g \circ f$ is continuous at $x_0$.*

## 4.2 Extension by continuity

**Definition 13.** *Let $I$ be an interval, $x_0$ a point in $I$, and $f : I \setminus \{x_0\} \longrightarrow \mathbb{R}$ a function.*

- *It is said that $f$ is extendable by continuity at $x_0$ if $f$ has a finite limit at $x_0$. We then denote the limit as $\lim\limits_{x \longrightarrow x_0} f(x) = \ell$.*
- *We then define the function $\tilde{f} : I \longrightarrow \mathbb{R}$ by setting for all $x \in I$.*

$$\tilde{f}(x) = \begin{cases} f(x) & if \ x \neq x_0 \\ \ell & if \ x = x_0. \end{cases}$$

*Then $\tilde{f}$ is continuous at $x_0$, and it is called the continuity extension of $f$ at $x_0$*

**Example 10.** *The function*

$$f(x) = x \sin\left(\frac{1}{x}\right)$$

*is defined and continuous on $\mathbb{R}^*$. Moreover, for all $x \in \mathbb{R}$, we have*

$$|f(x)| = \left| x \sin\left(\frac{1}{x}\right) \right| \leq |x|$$

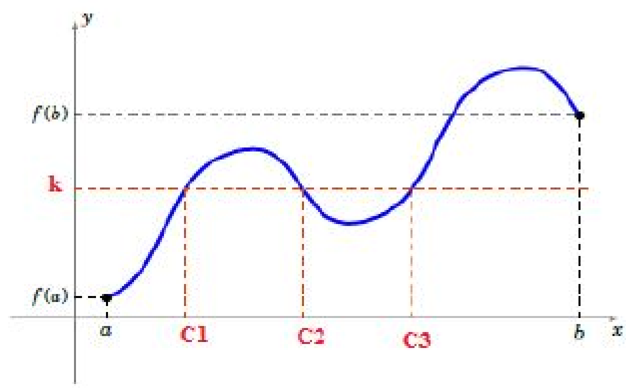
*So, $\lim\limits_{x \to 0} f(x) = 0$. The continuous extension of $f$ at point $0$ is therefore the function $\tilde{f}$ defined by*

$$\tilde{f}(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \textit{if } x \neq 0, \\ 0 & \textit{if } x = 0. \end{cases}$$

### 4.3 Intermediate Value Theorem

**Theorem 1.** *Let $f$ be a continuous function on the interval $[a, b]$*

*For every real number $k$ between $f(a)$ and $f(b)$, there exists $c \in [a, b]$ such that $f(c) = k$.*




Here is the most commonly used version of the Intermediate Value Theorem.

**Theorem 2.** *Let $f$ be a continuous function on the interval $[a, b]$*

*If $f(a) \cdot f(b) < 0$, then there exists $c \in ]a, b[$ such that $f(c) = 0$ .*

# 5 Derivative

### 5.1 Derivative at a point

Let $I$ be an open interval in $\mathbb{R}$, and let $f : I \longrightarrow \mathbb{R}$ be a function. Let $x_0$ be in $I$.

**Definition 14.** *$f$ is differentiable at $x_0$ if the rate of change $\frac{f(x)-f(x_0)}{x-x_0}$ has a finite limit as $x$ approaches $x_0$.*
*The limit is then called the derived number of $f$ at $x_0$ and is denoted as $f'(x_0)$. Thus*

$$\boxed{f'(x_0) = \lim_{x \longrightarrow x_0} \frac{f(x)-f(x_0)}{x-x_0}.}$$

**Remark 1.** *Another representation of the derivative is as follows:*

$$f \text{ is differentiable at } x_0 \text{ if and only if } \lim_{h \longrightarrow 0} \tfrac{f(x_0+h)-f(x_0)}{h} \text{ exists and is finite.}$$

**Definition 15.** *$f$ is differentiable on $I$ if $f$ is differentiable at every point $x_0 \in I$. The function $x \longmapsto f'(x)$ is the derivative function of $f$, it is denoted as $f'$ or $\frac{df}{dx}$.*

**Example 11.** *The function defined by $f(x) = x^2$ is differentiable at every point $x_0 \in \mathbb{R}$. Indeed*

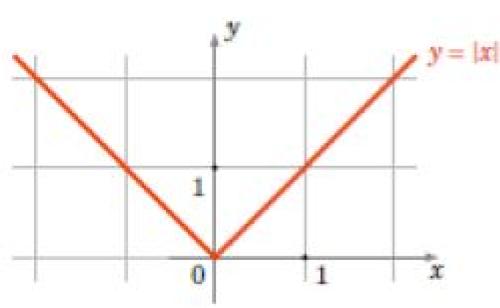
$$\frac{f(x)-f(x_0)}{x-x_0}. = \frac{x^2-x_0^2}{x-x_0} = \frac{(x-x_0)(x+x_0)}{x-x_0} = x + x_0 \underset{x \to x_0}{\longrightarrow} 2x_0.$$

*It has even been shown that the derived number of $f$ at $x_0$ is $2x_0$, in other words: $f'(x) = 2x$.*

**Definition 16.**
- *$f$ is right-differentiable at $x_0$, if $\lim\limits_{\substack{x \longrightarrow x_0 \\ >}} \frac{f(x)-f(x_0)}{x-x_0} = f_r'(x_0)$.*
- *$f$ is left-differentiable at $x_0$, if $\lim\limits_{\substack{x \longrightarrow x_0 \\ <}} \frac{f(x)-f(x_0)}{x-x_0} = f_l'(x_0)$.*
- *$f$ is differentiable at $x_0 \Longleftrightarrow f$ is right-differentiable and left-differentiable at $x_0$ and $f'(x_0) = f_r'(x_0) = f_l'(x_0)$.*

**Proposition 9.** *Let $I$ be an open interval, $x_0 \in I$, and let $f : I \longrightarrow \mathbb{R}$ be a function.*
- *If $f$ is differentiable at $x_0$, then $f$ is continuous at $x_0$.*
- *If $f$ is differentiable on $I$, then $f$ is continuous on $I$.*

**Remark 2.** *The converse is false: for example, the absolute value function is continuous at 0 but is not differentiable at 0.*



*Indeed, the rate of change of $f(x) = |x|$ at $x_0 = 0$, satisfies:*

$$\frac{f(x) - f(0)}{x - 0}. = \frac{|x|}{x} = \begin{cases} +1 & if \quad x > 0 \\ -1 & if \quad x < 0 \end{cases} .$$

*There is indeed a right-hand limit ($f_r^{'}(0) = +1$) and a left-hand limit ($f_l^{'}(0) = -1$), but they are not equal: there is no limit at 0. Therefore, $f$ is not differentiable at $x = 0$. This can also be seen in the graph; there is a half-tangent on the right and a half-tangent on the left, but they have different directions.*

**Proposition 10.** *Let $f, g : I \longrightarrow \mathbb{R}$ be two differentiable functions on $I$. Then*

- $(f + g)^{'} = f^{'} + g^{'}$
- $(\lambda f)^{'} = \lambda f^{'}$ *where $\lambda$ is a fixed real number*
- $(f \times g)^{'} = f^{'} g + f g^{'}$
- $\left(\frac{1}{f}\right)^{'} = -\frac{f^{'}}{f^2}$ *(if $f \neq 0$)*
- $\left(\frac{f}{g}\right)^{'} = \frac{g f^{'} - f g^{'}}{g^2}$ *(if $g \neq 0$)*

## 5.2 Derivative of common functions

$u$ represents a function $x \longmapsto u(x)$

| Function | Derivative |
| --- | --- |
| $x^n$ | $nx^{n-1} \quad (n \in \mathbb{Z})$ |
| $\frac{1}{x}$ | $-\frac{1}{x^2}$ |
| $\sqrt{x}$ | $\frac{1}{2\sqrt{x}}$ |
| $x^\alpha$ | $\alpha x^{\alpha-1} \quad (\alpha \in \mathbb{R})$ |
| $e^x$ | $e^x$ |
| $\ln x$ | $\frac{1}{x}$ |
| $\cos x$ | $-\sin x$ |
| $\sin x$ | $\cos x$ |
| $\tan x$ | $1 + \tan^2 x = \frac{1}{\cos^2 x}$ |

| Function | Derivative |
| --- | --- |
| $u^n$ | $nu^{'}u^{n-1} \quad (n \in \mathbb{Z})$ |
| $\frac{1}{u}$ | $-\frac{u^{'}}{u^2}$ |
| $\sqrt{u}$ | $\frac{u^{'}}{2\sqrt{u}}$ |
| $u^\alpha$ | $\alpha u^{'}u^{\alpha-1} \quad (\alpha \in \mathbb{R})$ |
| $e^u$ | $u^{'}e^u$ |
| $\ln u$ | $\frac{u^{'}}{u}$ |
| $\cos u$ | $-u^{'}\sin u$ |
| $\sin u$ | $u^{'}\cos u$ |
| $\tan u$ | $u^{'}(1 + \tan^2 u) = \frac{u^{'}}{\cos^2 u}$ |

## 5.3 Composition

**Proposition 11.** *If $f$ is differentiable at $x_0$ and $g$ is differentiable at $f(x_0)$, then $g \circ f$ is differentiable at $x_0$ with derivative:*

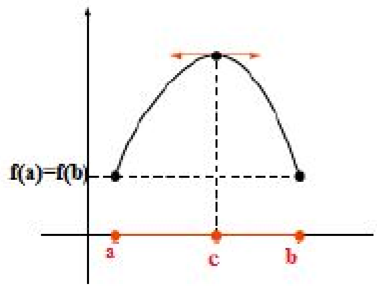
$$\boxed{(g \circ f)^{'}(x_0) = g^{'}(f(x_0)) \cdot f^{'}(x_0)}$$

## 5.4 Rolle's Theorem

**Theorem 3** (Rolle's Theorem)**.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ such that*

- *$f$ is continuous on $[a, b]$,*
- *$f$ is differentiable on $]a, b[$,*
- *$f(a) = f(b)$.*

*Then, there exists $c \in ]a, b[$ such that $f'(c) = 0$.*




*There exists at least one point on the graph of $f$ where the tangent is horizontal.*

## 5.5 Mean Value Theorem

**Theorem 4** (Mean Value Theorem)**.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function that is continuous on $[a, b]$ and differentiable on $]a, b[$.*
*Then, there exists $c \in ]a, b[$ such that*

$$\boxed{f(b) - f(a) = f'(c)(b - a)}$$

## 5.6 Increasing Function and Derivative

**Corollary 1.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function that is continuous on $[a, b]$ and differentiable on $]a, b[$.*

1. $\forall x \in ]a, b[ \quad f'(x) \geq 0 \quad \Longleftrightarrow \quad f$ *is increasing;*
2. $\forall x \in ]a, b[ \quad f'(x) \leq 0 \quad \Longleftrightarrow \quad f$ *is decreasing;*
3. $\forall x \in ]a, b[ \quad f'(x) = 0 \quad \Longleftrightarrow \quad f$ *is constant;*
4. $\forall x \in ]a, b[ \quad f'(x) > 0 \quad \Longrightarrow \quad f$ *is strictly increasing;*
5. $\forall x \in ]a, b[ \quad f'(x) < 0 \quad \Longrightarrow \quad f$ *is strictly decreasing.*

**Corollary 2** (L'Hôpital's Rule)**.** *Let $f, g : I \longrightarrow \mathbb{R}$ be two differentiable functions, and let $x_0 \in I$. We assume that*

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 \ (or\ \infty)$$

$$\boxed{\text{If} \quad \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \ell \ (\in \mathbb{R}) \qquad \text{then} \qquad \lim_{x \to x_0} \frac{f(x)}{g(x)} = \ell.}$$

**Example 12.** *Calculate the limit at 1 of* $\frac{\ln(x^2+x-1)}{\ln(x)}$. *We verify that:*

- $f(x) = \ln(x^2 + x - 1), \quad \lim_{x \to 1} f(x) = 0, \quad f^{'}(x) = \frac{2x+1}{x^2+x-1},$
- $f(x) = \ln(x), \quad \lim_{x \to 1} g(x) = 0, \quad g^{'}(x) = \frac{1}{x},$

$$\frac{f'(x)}{g'(x)} = \frac{2x+1}{x^2+x-1} \times x = \frac{2x^2+x}{x^2+x-1} \underset{x \to 1}{\longrightarrow} 3.$$

*Therefore*

$$\frac{f(x)}{g(x)} \underset{x \to 1}{\longrightarrow} 3.$$