Course of Mathematics1

Chapter 4 : An application of elementary functions

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CHAPITRE 1

An application of elementary functions

1.1 Logarithm function, exponential function and power function

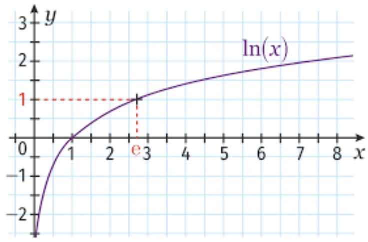
1.1.1 Logarithm function

Proposition 1.1.1 *There exist a unique function, denoted ln* $:]0,+\infty[\to \mathbb{R}$ *such that :*

$$(\ln x)' = \frac{1}{x}, \ \textit{for all } x, \ \textit{and } \ln(1) = 0.$$

1. $\ln(a \times b) = \ln a + \ln b$,
2. $\ln(\frac{a}{b}) = \ln a - \ln b$,
3. $\ln(\frac{1}{a}) = -\ln a$,
4. $\ln(a^n) = n \ln a$, $(\textit{for all } n \in \mathbb{N})$,
5. *The function* $x \to \ln(x)$ *is a continuous, strictly increasing function and*

defines a bijection of $]0, +\infty[$ *on* $\mathbb{R}$.



Particular limits :

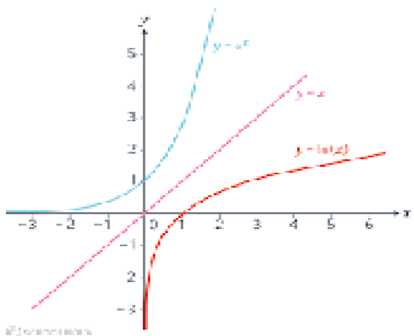
$$\lim_{x \to +\infty} \ln(x) = +\infty, \quad \lim_{x \to 0^+} \ln(x) = -\infty, \quad \lim_{x \to 0^+} x \ln(x) = 0,$$

$$\lim_{x \to +\infty} \frac{\ln(x)}{x} = 0, \qquad \lim_{x \to 0^+} \frac{\ln(x+1)}{x} = 1.$$

1.1.2 Exponential function

Definition 1.1.1 *The inverse function of* $\ln :]0, +\infty[\to \mathbb{R}$ *is called the exponential function, noted* $\exp : \mathbb{R} \to]0, +\infty[$.

For $x \in \mathbb{R}$, *we also note* e^x *for* $\exp x$.



Proposition 1.1.2 *The exponential function satisfies the following properties :*

1. $e^{(\ln x)} = x, \forall x \in \mathbb{R}$, *and* $\ln(e^x) = x, \forall x \in \mathbb{R}$,
2. $e^{(a+b)} = e^a \times e^b$,
3. $(e^a)^n = e^{na}$,
4. $e^{-a} = \frac{1}{e^a}$,
5. $\exp : \mathbb{R} \to]0, +\infty[$ *is a continuous function, strictly increasing.*
6. *The exponential function is differentiable and* $(e^x)' = e^x$, *for all* $x \in \mathbb{R}$.

Particular limits :
$\lim\limits_{x\to+\infty} e^x = +\infty$, $\lim\limits_{x\to-\infty} e^x = 0$, $\lim\limits_{x\to+\infty} xe^x = +\infty$,
$\lim\limits_{x\to+\infty} \frac{e^x}{x} = +\infty$, $\lim\limits_{x\to 0} \frac{e^x-1}{x} = 1$.

1.1.3 Power function

Definition 1.1.2 *The* ***power function*** *of a positive real a is the function f_a defined on $\mathbb{R}$ by :*

$$f_a(x) = a^x = e^{x\ln a}.$$

Remark 1.1.1 *The power function is strictly positive,*

$$\forall x \in \mathbb{R}, a^x = e^{x\ln a} > 0.$$

Properties

For all $a, b > 0$, we have the following equalities :

1. $\forall x, y \in \mathbb{R}$, $a^{x+y} = a^x \times a^y$ and $a^{x-y} = \frac{a^x}{a^y}$,
2. $\forall x, y \in \mathbb{R}$, $(a^x)^y = a^{xy}$,
3. $\forall x \in \mathbb{R}$, $(ab)^x = a^x \times b^x$.

Study of the power function

Let be the function f_a defined on $\mathbb{R}$ by : $f_a(x) = a^x$:

As $a^x = e^{x\ln a}$, it is continuous and differentiable on $\mathbb{R}$; because it is a composition of continuous and differentiable functions on $\mathbb{R}$. So :

$$\forall x \in \mathbb{R},\ f_a'(x) = (e^{x\ln a})' = \ln(a)e^{x\ln a} = \ln(a)a^x.$$

The sign of the derivative therefore depends on the sign of $\ln(a)$. We have then :

1. If $a > 1$, then $\forall x \in \mathbb{R},\ f_a'(x) > 0$, the power function is increasing.
2. If $0 < \alpha < 1$, then $\forall x \in \mathbb{R},\ f_a'(x) < 0$, the power function is decreasing.

Particular limits :

$a > 1$	$0 < \alpha < 1$
$\lim\limits_{x \to +\infty} a^x = +\infty$	$\lim\limits_{x \to +\infty} a^x = 0$
$\lim\limits_{x \to -\infty} a^x = 0$	$\lim\limits_{x \to -\infty} a^x = +\infty$

1.2 Trigonometric functions

1.2.1 Sine and cosine functions

Function	$\sin x$	$\cos x$
Definition domain	$\mathbb{R}$	$\mathbb{R}$
Parity	Odd	even
Period	$T = 2\pi$	$T = 2\pi$
Derivative	$\cos x$	$-\sin x$

Properties :

The **sine** and **cosine** functions satisfy the following properties, for all $x \in \mathbb{R}$:

- $\cos^2 x + \sin^2 x = 1$,
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$,
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$,
- $\sin 2x = 2 \sin x \cos x$,
- $\cos 2x = \cos^2 x - \sin^2 x = -1 + 2\cos^2 x = 1 - 2\sin^2 x$.

Addition formula

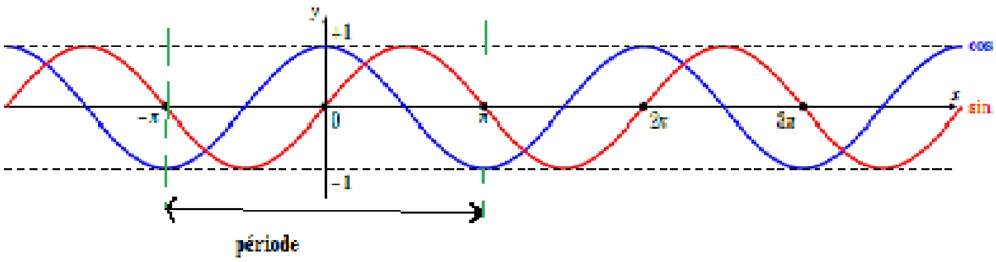
$\forall a, b \in \mathbb{R}$, we have :

$$\begin{aligned}
\sin(a+b) &= \sin a \cos b + \sin b \cos a \\
\sin(a-b) &= \sin a \cos b - \sin b \cos a \\
\cos(a+b) &= \cos a \cos b - \sin a \sin b \\
\cos(a-b) &= \cos a \cos b + \sin a \sin b
\end{aligned}$$

variations

The sine and cosine functions are continuous and differentiable on all $\mathbb{R}$. As they are periodic, of period 2π, we can restrict the domain of the study to the interval of length 2π, for example $[-\pi, \pi]$.

cos



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1.2.2 The tangent and cotangent functions

Definition 1.2.1 *We call* ***tangent*** *the function* tan *(or tg) defined by :*

$$x \longmapsto \tan x = \frac{\sin x}{\cos x},\ for\ all\ x \in \mathbb{R} - A$$

where $A = \left\{ \frac{\pi}{2} + k\pi / \ k \in \mathbb{Z} \right\}$.

We call **cotangent** the function cot defined by :

$$x \longmapsto \cot x = \frac{\cos x}{\sin x},\ for\ all\ x \in \mathbb{R} - B$$

where $B = \{k\pi / \ k \in \mathbb{Z}\}$.

For all $x \in \mathbb{R} - \{A \cup B\}$, we have the equality $\cot x = \frac{1}{\tan x}$.

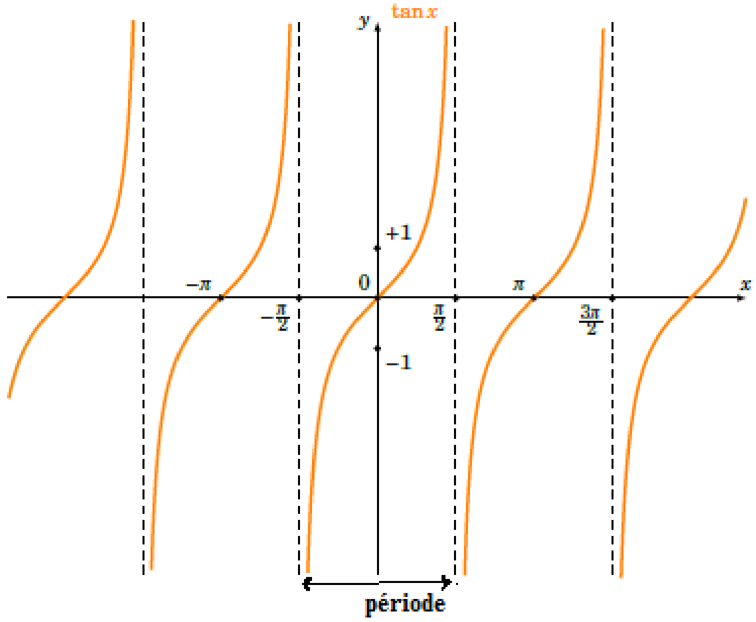
Derivatives and variations

The tangent and cotangent functions are continuous and differentiable over their defined domains and we have :

$$\begin{aligned}(\tan x)' &= \frac{1}{\cos^2 x} = 1 + \tan^2 x,\ \forall\ x \in \mathbb{R} - A \\ (\cot x)' &= -\frac{1}{\sin^2 x} = -(1 + \cot^2 x),\ \forall x \in \mathbb{R} - B.\end{aligned}$$

The two functions being periodic with period π , we can therefore restrict the domain of the study to an interval of length π, for example $\left]\frac{-\pi}{2}, \frac{\pi}{2}\right[$for the tangent and $]0, \pi[$ for the

cotangent.



1.2.3 Inverse trigonometric functions

Arccosine

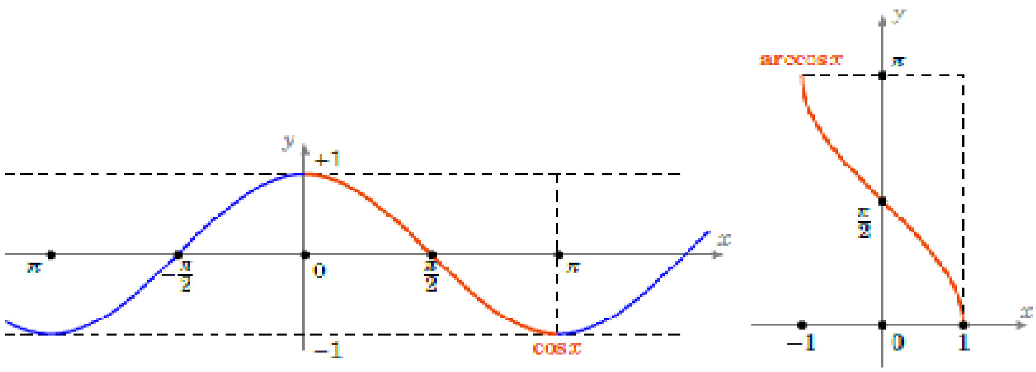
Consider the cosine function $\cos : \mathbb{R} \to [-1, 1]\,,\ x \mapsto \cos x$.

To obtain a bijection from this function, we must consider the restriction of cosine at interval $[0, \pi]$. On this interval the cosine function is continuous and strictly decreasing, therefore

$$\cos : [0, \pi] \longrightarrow [-1, 1]$$

is a bijection. Its inverse function (bijection) is the **arccosine** function :

$$\arccos [-1, 1] :\longrightarrow [0, \pi].$$



Thus, by definition of inverse function :

$$\begin{aligned} \cos(\arccos x) &= x, \ \forall x \in [-1, 1]\,, \\ \arccos(\cos x) &= x, \ \forall x \in [0, \pi]. \end{aligned}$$

In other words :

$$If \quad x \in [0, \pi], \ \ \cos x = y \Leftrightarrow x = \arccos y.$$

The derivative of arccos :

$$(\arccos x)' = \frac{-1}{\sqrt{1 - x^2}}, \ \ \forall x \in \left]-1, 1\right[.$$

Proof We start from the equality $cos(\arccos x) = x$ that we derive :

$$\begin{aligned}
(cos(\arccos x) &= x)' \Rightarrow -(\arccos x)' \times \sin(\arccos x) = 1 \\
&\Rightarrow (\arccos x)' = \frac{-1}{\sin(\arccos x)} \\
&\Rightarrow (\arccos x)' = \frac{-1}{\sqrt{1-\cos^2(\arccos x)}} \qquad (*) \\
&\Rightarrow (\arccos x)' = \frac{-1}{\sqrt{1-x^2}}
\end{aligned}$$

The crucial point $(*)$ is justified as follows : we start from equality $\cos^2\alpha + \sin^2\alpha = 1$, by substituting $\alpha = \arccos x$, we get

$$\cos^2(\arccos x) + \sin^2(\arccos x) = 1.$$

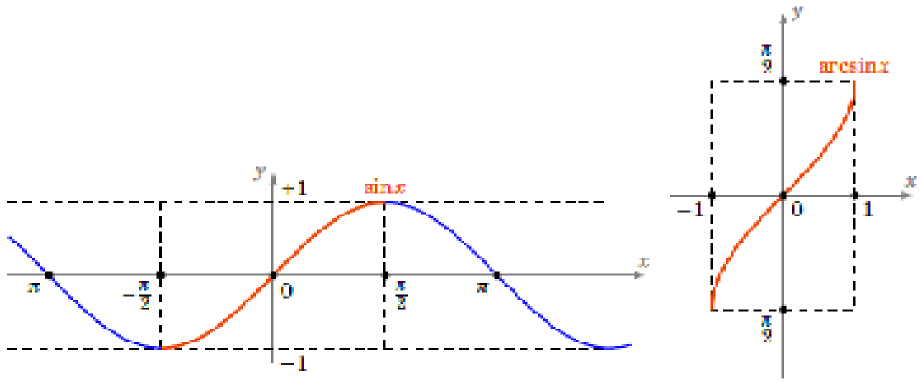
So, $x^2 + \sin^2(\arccos x) = 1$.This leads to : $\sin\arccos x = +\sqrt{1-x^2}$ (with the + sign because $\arccos x \in [0, \pi]$, and so we have $\sin(\arccos x \geq 0)$. ■

Arcsine

The sine function has a strictly positive derivative function on $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, so it is a

bijection of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ on $[-1, 1]$. The inverse of sine function is called an **arcsine** function and is denoted arcsin,

$$\begin{aligned}
\arcsin \quad &: \quad [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\
x \quad &\mapsto \quad \arcsin x.
\end{aligned}$$



Properties :

1. $\sin(\arcsin x) = x, \, \forall x \in [-1, 1]$,
2. $\arcsin(\sin x) = x, \, \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,
3. $\sin x = y \Leftrightarrow x = \arcsin y, \, \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,
4. The arcsin function is derivable on $]-1, 1[$, and we have

$$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}.$$

Arctangent

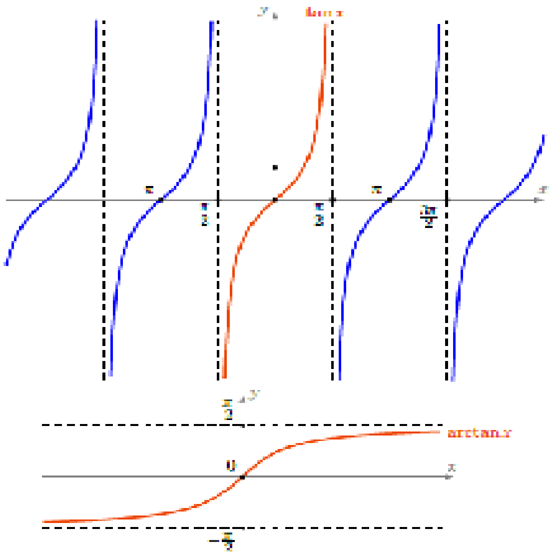
The function

$$\tan :]-\frac{\pi}{2}, +\frac{\pi}{2}[\to \mathbb{R}$$

is a bijection.

Its inverse function is the function **arctangent** :

$$\arctan : \mathbb{R} \to]-\tfrac{\pi}{2}, +\tfrac{\pi}{2}[$$



Properties :

1. $\tan(\arctan(x)) = x, \forall x \in \mathbb{R}$,
2. $\arctan(\tan(x)) = x, \forall x \in]-\frac{\pi}{2}, +\frac{\pi}{2}[$,
3. $\tan(x) \ = y \Leftrightarrow x = \arctan\, y,\ \forall x \in]-\frac{\pi}{2}, +\frac{\pi}{2}[$,
4. The arctan function is derivable on $\mathbb{R}$, and we have

$$(\arctan(x))' = \frac{1}{1+x^2}$$

1.3 Hyperbolic and inverse hyperbolic functions

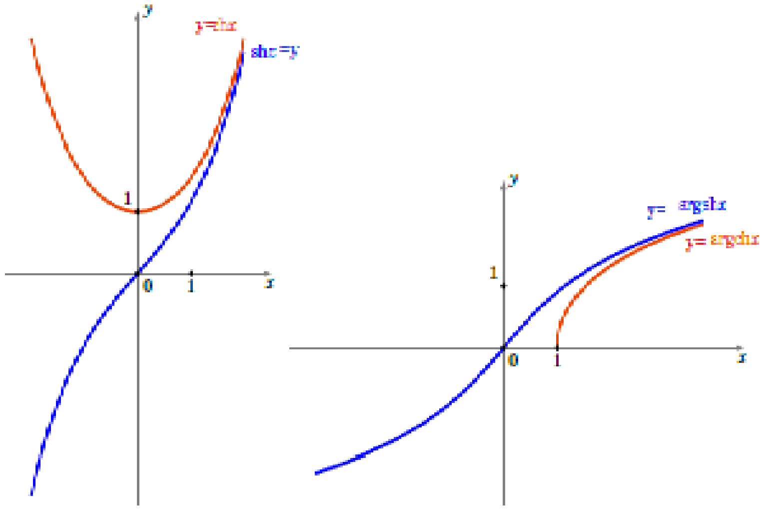
1.3.1 Hyperbolic cosine and its inverse

For $x \in \mathbb{R}$, the hyperbolic cosine is :

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

The function $\cosh : [0, +\infty[\to [1, +\infty[$ is a bijection.

Its inverse function is $Argch : [1, +\infty[\to [0, +\infty[$. (Hyperbolic cosine argument)



1.3.2 Hyperbolic sine and its inverse

For $x \in \mathbb{R}$, the hyperbolic sine is :

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$\sinh : \mathbb{R} \to \mathbb{R}$ is a continuous, differentiable, strictly increasing function verifying

$$\lim_{x \to +\infty} \sinh x = +\infty \text{ and } \lim_{x \to -\infty} \sinh x = -\infty,$$

therefore, sinh is a bijection.

The inverse function is $Argsh : \mathbb{R} \to \mathbb{R}$. (Hyperbolic sine argument)

Proposition 1.3.1 *1.* $\cosh^2 x - \sinh^2 x = 1$.

2. $(\cosh x)' = \sinh x$, $(\sinh x)' = \cosh x$.

3. *$Argsh : \mathbb{R} \to \mathbb{R}$ is strictly increasing and continuous.*

4. *$Argsh$ is differentiable and $(Argshx)' = \frac{1}{\sqrt{x^2+1}}$.*

5. $Argsh\ x$ $=\ln(x+\sqrt{x^2+1})$.

Proof

1. $\cosh^2 x - \sinh^2 x = \frac{1}{4}\left[(e^x+e^{-x})^2 - (e^x-e^{-x})^2\right]$
   $= \frac{1}{4}\left[(e^{2x}+2+e^{-2x}) - (e^{2x}-2+e^{-2x})\right] = 1$
2. $\frac{d}{dx}(\cosh x) = \frac{d}{dx}\frac{e^x+e^{-x}}{2} = \frac{e^x-e^{-x}}{2} = \sinh x$
3. Because it is the inverse of sinh.
4. Because the function $x \to \acute{\sinh}\, x$ does not cancel on $\mathbb{R}$ then the function $Argsh$ is differentiable on $\mathbb{R}$. We calculate the derivative by deriving the equality $\sinh(Argsh\ x) = x$:

$$\begin{aligned}(Argsh\ x)' &= \frac{1}{\cosh(Argsh\ x)} = \frac{1}{\sqrt{\sinh^2(Argsh\ x)+1}} \\ &= \frac{1}{\sqrt{x^2+1}}\end{aligned}$$

5. Note $f(x) = \ln(x+\sqrt{x^2+1})$ so

$$\acute{f}(x) = \frac{1+\frac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}} = (Argsh\ x)'$$

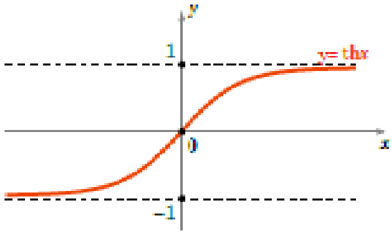
As more $f(0) = \ln(1) = 0$ and $Argsh\ 0 = 0$ (because $\sinh 0 = 0$), we deduce that for all $x \in \mathbb{R}$, $f(x) = Argsh\ x$.

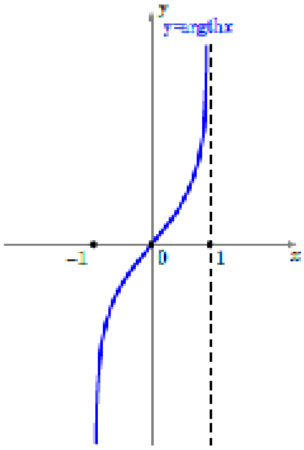
1.3.3 Hyperbolic tangent and its inverse

By definition the hyperbolic tangent is :

$$\tanh x = \frac{\sinh x}{\cosh x}$$

The tanh function : $\mathbb{R} \to]-1,1[$ is a bijection, we note $Argth :]-1,1[\to \mathbb{R}$ its inverse function.





1.3.4 Hyperbolic trigonometry

- $\cosh^2 x - \sinh^2 x = 1$
- $\cosh(a+b) = \cosh a \cdot \cosh b + \sinh a \cdot \sinh b$
  $\cosh(2a) = \cosh^2 a + \sinh^2 a = 2\cosh^2 a - 1 = 1 + 2\sinh^2 a$
- $\sinh(a+b) = \sinh a \cdot \cosh b + \sinh b \cdot \cosh a$
  $\sinh(2a) = 2\sinh a \cdot \cosh a.$
- $\tanh(a+b) = \frac{\tanh a + \tanh b}{1 + \tanh a \cdot \tanh b}$
- $(\cosh x)' = \sinh x$
  $(\sinh x)' = \cosh x$
  $(\tanh x)' = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}$
- $(Argch\ x)' = \frac{1}{\sqrt{x^2-1}} \quad (x > 1)$
  $(Argsh\ x)' = \frac{1}{\sqrt{x^2+1}}$
  $(Argth\ x)' = \frac{1}{1-x^2} \quad (|x| < 1)$
- $Argch\ x = \ln(x + \sqrt{x^2-1}) \quad (x \geqslant 1)$
  $Argsh\ x = \ln(x + \sqrt{x^2+1}) \quad (x \in \mathbb{R})$
  $Argth\ x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad (-1 < x < 1).$