Course of Maths I
Chapter VI: LIMITED DEVELOPMENT

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Contents

In this chapter, for any given function, we will find the polynomial of degree n that best approximates the function around a specific point. More precisely, we aim to decompose any sufficiently regular function f around a given point a as follows:

$$f(x) = T_n(x) + R_n(x);$$

where T_n is a polynomial of degree n and R_n is a function that satisfies $\lim_{x\to a} R_n(x) = 0$. Decomposing a function in this way around a point a is known as performing a limited development (LD) of the function at point a to order n. The polynomial T_n is called the polynomial part of the LD, and the function R_n is referred to as the remainder of the expansion.

1 Limited developments

1.1 Notations

Let $I \subset \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ a function, and $n \in \mathbb{N}^*$. - If f is differentiable and if the function $f' : I \to \mathbb{R}$ is also differentiable, we denote $f'' = (f')'$ as the second derivative of f. More generally, for any $n \in \mathbb{N}$, we denote

$$f^{(0)} = f, f^{(1)} = f', f^{(2)} = f'', \dots, f^{(n+1)} = (f^{(n)})'$$

.
- f is of class C^0 on I if f is continuous on I. We write $f \in C^0(I)$.
- f is of class C^n on I if f is n-times differentiable on I and $f^{(n)}$ is continuous on I. We write $f \in C^n(I)$.
- f is of class C^∞ on I if $\forall n \in \mathbb{N}, f \in C^n(I)$. We write $f \in C^\infty(I)$. Note that for any $n \in \mathbb{N}$, $C^\infty(I) \subset C^n(I)$.
- For any $n \in \mathbb{N}^*$, the factorial of n is defined by $n! = 1 \times 2 \times \ldots \times (n-1) \times n$ with the convention $0! = 1$.

1.2 Definition and existence

Let I be an open interval and $f : I \to \mathbb{R}$ a function.

Definition 1. *For $a \in I$ and $n \in \mathbb{N}$, we say that f admits a limited development at point a to order n if there exist $n+1$ real constants $c_0, c_1, \dots, c_n$ and a function $\varepsilon : I \to \mathbb{R}$ such that $\lim_{x\to a} \varepsilon(x) = 0$ and, for every x in I,*

$$f(x) = \underbrace{c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n}_{\textit{Polynomial part of the LD}} + \underbrace{(x-a)^n\varepsilon(x)}_{\textit{Remainder of the LD}}.$$

Proposition 1. *Let I be an open interval and $a \in I$. If f is of class C^n on I, then f admits a limited development of order n at the point a given by the following formula known as the Taylor-Young formula:*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + (x-a)^n\varepsilon(x),$$

where $\lim_{x\to a} \varepsilon(x) = 0$.

Remark 1. *The limited development of a function f of order n at the point 0 is written as follows using the Taylor-Young formula:*

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + x^n\epsilon(x)$$

where $\lim_{x\to 0} \varepsilon(x) = 0$.

Example 1. *Let $a \in \mathbb{R}$ and $n \in \mathbb{N}^*$. Give the limited development at the point a to order n of the function $f(x) = e^x$.*
$f(x) = e^x, \quad f'(x) = e^x, \quad f''(x) = e^x, \quad \ldots, \quad f^{(n)}(x) = e^x.$
Evaluating each at $x = a$, we obtain:
$f(a) = e^a, \quad f'(a) = e^a, \quad f''(a) = e^a, \quad \ldots, \quad f^{(n)}(a) = e^a.$
Thus, the limited development of e^x at $x = a$ to order n is:

$$\begin{aligned} e^x &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + (x-a)^n\varepsilon(x). \\ &= e^a + e^a(x-a) + \frac{e^a}{2!}(x-a)^2 + \cdots + \frac{e^a}{n!}(x-a)^n + (x-a)^n\varepsilon(x). \\ &= e^a\left(1 + (x-a) + \frac{(x-a)^2}{2!} + \cdots + \frac{(x-a)^n}{n!}\right) + (x-a)^n\varepsilon(x), \end{aligned}$$

where $\lim_{x\to 0} \varepsilon(x) = 0$.

Example 2. *Give the limited development of $f(x) = \sin x$ of order 5 at the point 0.*
$f(x) = \sin x, f'(x) = \cos x, f''(x) = -\sin x, f^{(3)}(x) = -\cos x, f^{(4)}(x) = \sin x, f^{(5)}(x) = -\cos x$
Evaluating each at $x = a$, we obtain:
$f(0) = 0, f'(0) = 1, f''(0) = 0, f^{(3)}(0) = -1, f^{(4)}(0) = 0, f^{(5)}(0) = 0.$
Then

$$\begin{aligned} \sin(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(5)}(0)}{5!}x^5 + x^6\varepsilon(x), \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + x^6\varepsilon(x). \end{aligned}$$

where $\lim_{x\to 0} \varepsilon(x) = 0$.

Notations

1. The term $(x-a)^n\varepsilon(x)$ where $\varepsilon(x) \to 0$ as $x \to a$ is often denoted as $o_a((x-a)^n)$, this means that the function $x \mapsto o_a((x-a)^n)$ satisfies the property:

$$\lim_{x\to a} \frac{o_a((x-a)^n)}{(x-a)^n} = 0.$$

2. The remainder $x^n\varepsilon(x)$ of limited development of a function f at the point 0 of order n can also be written as $x^n\varepsilon(x) = o(x^n)$.

Proposition 2. *If a function f has a limited development at a point, then this limited development is unique.*

Proposition 3. - *If f is an even function, then the polynomial part of its limited development at 0 contains only monomials of even degrees (i.e., $x^0, x^2, x^4, \ldots, x^{2n}, n \in \mathbb{N}$).*
- If f is an odd function, then the polynomial part of its limited development at 0 contains only monomials of odd degrees (i.e., $x, x^3, x^5, \ldots, x^{2n+1}, n \in \mathbb{N}$).

Example 3. *On considère la fonction $f(x) = \cos(x)$, qui est de classe $C^\infty(\mathbb{R})$. It admits a limited development at 0 of order 5.*
$f(x) = \cos(x)$, $\quad f'(x) = -\sin(x)$, $\quad f''(x) = -\cos(x)$, $\quad f'''(x) = \sin(x)$, $\quad f^{(4)}(x) = \cos(x)$, $\quad f^{(5)}(x) = -\sin(x)$.
Evaluating each at $x = 0$, we obtain: $f(0) = 1$, $\quad f'(0) = 0$, $\quad f''(0) = -1$, $\quad f'''(0) = 0$, $\quad f^{(4)}(0) = 1$, $\quad f^{(5)}(0) = 0$.
Thus, the limited development of $\cos(x)$ at $x = 0$ to order 5 is:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + o(x^6)$$

By substituting the calculated values :

$$f(x) = 1 + 0 \cdot x + \frac{-1}{2!}x^2 + 0 \cdot x^3 + \frac{1}{4!}x^4 + 0 \cdot x^5 + o(x^6) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^6)$$

Since f is an even function, the polynomial part of its Taylor expansion only contains terms of even degrees, as stated in the proposition.

1.3 The Limited development of Common functions at 0 up to Order n

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + o(x^n)$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n+1}\frac{x^n}{n} + o(x^n)$
- $\forall \alpha \in \mathbb{R}, \quad (1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n + o(x^n)$
- $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + o(x^n)$
- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + o(x^n)$
- $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$
- $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$
- $\tan(x) = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + o(x^8)$
- $\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$
- $\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$
- $\tanh(x) = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + o(x^8)$

Proposition 4. *A function f has a limited development near to a point a if and only if the function $f(x+a)$ has a limited development near to a point 0.*

Example 4. *Calculate the $LD_n(1)$ of the function $f(x) = e^x$.*
We pose $h = x - 1$, if x is near to 1, then h is near to 0. Then

$$\begin{aligned}
e^x &= e^{x-1+1} \\
&= e^1 \times e^{x-1} \\
&= e \times e^h \\
&= e\left(1 + h + \frac{h^2}{2!} + \dots + \frac{h^n}{n!} + h^n\varepsilon(x)\right) \\
&= e + \frac{e}{1!}(x-1) + \frac{e}{2!}(x-1)^2 + \dots + \frac{e}{n!}(x-1)^n + o(x-1)^n,
\end{aligned}$$

2 Operations on limited developments

2.1 Sum and Product

Definition 2. *Let $n, p \in N^*$ with $n < p$. Truncating a polynomial of degree p to order n means keeping only the monomials of degrees $\leq n$.*

Example 5. *1. Truncate the polynomial $P(x) = x^8 + 2x^7 - 3x^5 + 2x^4 - x^3 + x^2 + 1$ to order 5. We will denote the obtained polynomial by $T_5(x)$.*

$$T_5(x) = 1 + x^2 - x^3 + 2x^4 - 3x^5$$

2. Truncate the polynomial $(3x+4x^2)(1+x+x^3)$ to order 2. We will denote the obtained polynomial by $T_2(x)$. We have:

$$(3x + 4x^2)(1 + x + x^3) = 3x(1 + x + x^3) + 4x^2(1 + x + x^3) = 3x + 7x^2 + 4x^3 + 3x^4 + 4x^5$$

Then

$$T_2(x) = 3x + 7x^2$$

Proposition 5. *Consider two functions f and g that admit limited development at 0 of order n:*

$$f(x) = c_0 + c_1x + \cdots + c_nx^n + x^n\varepsilon_1(x) \quad \textit{and} \quad g(x) = d_0 + d_1x + \cdots + d_nx^n + x^n\varepsilon_2(x),$$

with $\lim_{x\to 0}\varepsilon_1(x) = \lim_{x\to 0}\varepsilon_2(x) = 0$.
Then:

- *$f + g$ admits a limited development at 0 of order n , given by:*

$$f(x)+g(x)=(c_0+d_0)+(c_1+d_1)x+\cdots+(c_n+d_n)x^n+x^n\varepsilon(x),$$

where $\lim_{x\to 0}\varepsilon(x)=0$.

- $f\times g$ *admits a limited development at 0 of order* n *, given by:*

$$f(x)\times g(x)=T_n(x)+x^n\varepsilon(x),$$

where $\lim_{x\to 0}\varepsilon(x)=0$, *and* $T_n(x)$ *is the polynomial*

$$(c_0+c_1x+\cdots+c_nx^n)\times(d_0+d_1x+\cdots+d_nx^n)$$

truncated to order n.

Example 6. *Calculate the limited development at 0 of order 2 of the function* $f(x)=e^x+\ln(1+x)$. *The functions* e^x *and* $\ln 1+x$ *are* C^∞ *near 0, so we can write their limited development at 0 of order 2:*

$$e^x=1+x+\frac{x^2}{2!}+o(x^2)$$

and

$$\ln(1+x)=x-\frac{x^2}{2}+o(x^2).$$

then

$$f(x)=\left(1+x+\frac{x^2}{2!}+o(x^2)\right)+\left(x-\frac{x^2}{2}+o(x^2)\right)=1+2x+o(x^2)$$

Example 7. *Calculate the limited development at 0 of order 2 of the function* $f(x)=\cos(x)\times\sqrt{(1+x)}$.
The functions $\cos(x)$ *and* $\sqrt{1+x}$ *are* C^∞ *near 0, so we can write their limited development at 0 up to order 2:*

$$\cos(x)=1-\frac{x^2}{2!}+o(x^2)=1-\frac{x^2}{2}+o(x^2).$$

and

$$\sqrt{1+x}=(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2+o(x^2)$$

then

$$\left(1-\frac{x^2}{2}+o(x^2)\right)\times\left(1+\frac{1}{2}x-\frac{1}{8}x^2+o(x^2)\right)=1+\frac{1}{2}x-\frac{5}{8}x^2+o(x^2)$$

By truncating the result at order 2, we obtain

$$f(x)=1+\frac{1}{2}x-\frac{1}{8}x^2+o(x^2)$$

2.2 Composition

Proposition 6. *We consider two functions f and g that admit a limited development at 0 up to order n:*

$$f(x) = c_0 + c_1x + \cdots + c_nx^n = C(x) + x^n\varepsilon_1(x) \quad \text{and} \quad g(x) = d_0 + d_1x + \cdots + d_nx^n = D(x) + x^n\varepsilon_2(x),$$

with $\lim_{x\to 0} \varepsilon_1(x) = \lim_{x\to 0} \varepsilon_2(x) = 0$.
If g admits a limited development at 0 up to order n and f admits a at $g(0)$ up to order n, then $f \circ g$ admits a limited development at 0 up to order n. This expansion is obtained by substituting the limited development of g into that of f and retaining only terms of degree $\leq n$.

Example 8. *Let the functions $f(x) = e^x$ and $g(x) = \sin(x)$. Calculate the $LD_2(0)$ of the composition $f(g(x)) = e^{\sin(x)}$.*

The $LD_2(0)$ of $f(x)$ and $g(x)$ are

$$f(x) = e^x = 1 + x + \frac{x^2}{2} + o(x^2)$$

$$g(x) = x - \frac{x^3}{6} + o(x^3).$$

Then

$$f(g(x)) = e^{\sin(x)} = 1 + \left(x - \frac{x^3}{6} + o(x^3)\right) + \frac{1}{2}\left(x - \frac{x^3}{6} + o(x^3)\right)^2 + o\left(\left(x - \frac{x^3}{6} + o(x^3)\right)^2\right)$$

By truncating the result at order 2, we obtain

$$f(g(x)) = 1 + x - \frac{x^3}{6} + \frac{1}{2}\left(x^2 - \frac{x^4}{3} + o(x^4)\right) + o(x^2) = 1 + x + \frac{x^2}{2} + o(x^2)$$

2.3 Quotient

Consider two functions f and g that admit a limited development at 0 up to order n:

$$f(x) = c_0 + c_1x + \cdots + c_nx^n + x^n\varepsilon_1(x) \quad \text{and} \quad g(x) = d_0 + d_1x + \cdots + d_nx^n + x^n\varepsilon_2(x),$$

with $\lim_{x\to 0} \varepsilon_1(x) = \lim_{x\to 0} \varepsilon_2(x) = 0$.
To calculate the limited development of the quotient $\frac{f}{g}$, we will use the limited development of

$$\frac{1}{1+u} = 1 - u + u^2 - u^3 + \cdots + (-1)^n u^n + o(u^n),$$

We have three possible cases:
Case 1: If $d_0 = 1$, we set $u = d_1x + \cdots + d_nx^n + x^n\varepsilon_2(x)$, and the quotient can be written as

$$\frac{f}{g} = f \times \frac{1}{1+u}.$$

Case 2: If $d_0 \neq 0$ and $d_0 \neq 1$, then we reduce to the previous case by rewriting

$$\frac{1}{g(x)} = \frac{1}{d_0} \times \frac{1}{1 + \frac{d_1}{d_0}x + \cdots + \frac{d_n}{d_0}x^n + x^n\frac{\varepsilon_2(x)}{d_0}}.$$

Case 3: If $d_0 = 0$, then we factor by x^k (for some k) to reduce to one of the previous cases.

Example 9. *We aim to find the $LD_4(0)$ of the function $f(x) = \frac{x}{\sin(x)}$.*
The limited development of $\sin(x)$ *at* $x = 0$ *is given by:*

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$$

then

$$f(x) = \frac{x}{\sin(x)} = \frac{x}{x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)} = \frac{1}{1 - \frac{x^2}{6} + \frac{x^4}{120} + o(x^4)}$$

Using the expansion for $\frac{1}{1+u}$

$$\frac{1}{1+u} = 1 - u + u^2 + o(x^2)$$

where

$$u = -\frac{x^2}{6} + \frac{x^4}{120} + o(x^4).$$

We find

$$f(x) = 1 - \left(-\frac{x^2}{6} + \frac{x^4}{120}\right) + \left(-\frac{x^2}{6} + \frac{x^4}{120}\right)^2 + o(x^4)$$

By truncating the result at order 4*, we obtain*

$$f(x) = 1 - \left(-\frac{x^2}{6} + \frac{x^4}{120}\right) + \left(-\frac{x^2}{6}\right)^2 + o(x^4)$$

$$f(x) = 1 + \frac{x^2}{6} + \frac{11x^4}{120} + o(x^4)$$

2.4 Integration

Let F be a primitive of f. The function F has a limited development at a up to order $n+1$, which is written as:

$$F(x) = F(a) + c_0(x-a) + c_1\frac{(x-a)^2}{2!} + \cdots + c_n\frac{(x-a)^{n+1}}{(n+1)!} + (x-a)^{n+1}\theta(x)$$

where $\lim_{x\to a}\theta(x) = 0$.
This means that we integrate the polynomial part term by term to obtain the Taylor expansion of $F(x)$, starting from the constant $F(a)$.

Example 10. *Let* $f(x) = \arctan(x)$ *be defined on* $\mathbb{R}$*. Let's find its limited development at* 0*.*
We have $f'(x) = \frac{1}{1+x^2}$*, the* $LD_n(0)$ *of* $\frac{1}{1+x^2}$ *at* 0 *is:*

$$f'(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 + ... + (-1)^n x^{2n} + o(x^{2n})$$

Now, integrate $f'(x)$ *to find* $f(x)$*:*

$$f(x) = \int \left(1 - x^2 + x^4 + ... + (-1)^n x^{2n} + o(x^{2n})\right) dx$$

we get:

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + o(x^{2n+1})$$

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3 Applications of Limit Developments

3.1 Limit Calculation

We seek to calculate $\lim_{x\to a} f(x)$. If f admits a limited development at a to the order n, then

$$f(x) = c_0 + c_1(x-a) + \ldots + c_n(x-a)^n + (x-a)^n \varepsilon(x)$$

and therefore,

$$\lim_{x\to a} f(x) = \lim_{x\to a} \left(c_0 + c_1(x-a) + \ldots + c_n(x-a)^n + (x-a)^n \varepsilon(x)\right).$$

Example 11. *Calculate* $\lim_{x\to 0} \frac{\sin(x)-x}{x(\cos(x)-1)}$*.*
We can see that this limit is an indeterminate form. We know the $DL_3(0)$ *of* $\sin(x)$ *and* $\cos(x)$ *in* 0

$$\sin(x) - x = \tfrac{x^3}{3!} + x^3\varepsilon(x),$$
$$x(\cos(x) - 1) = \tfrac{x^3}{2!} + x^3\varepsilon(x),$$

by substituting we have

$$\lim_{x\to 0} \tfrac{\sin(x)-x}{x(\cos(x)-1)} = \lim_{x\to 0} \tfrac{\frac{x^3}{3!}}{\frac{x^3}{2!}} = \lim_{x\to 0} \tfrac{\frac{1}{6}}{\frac{1}{2}} = \tfrac{1}{3}.$$

3.2 Continuity and Differentiability from a limited development

Let $f(x)$ be a function defined on an interval I, except at $x = 0$. If f admits a first-order limited development at $x = 0$, meaning $f(x) = a_0 + a_1 x + x\epsilon(x)$; then the following results hold:
1. **Extension by Continuity**: we can extend the function to be continuous at $x = 0$ by defining: $f(0) = a_0$
2. **Differentiability at** $x = 0$: This extension guarantees that $f(x)$ is differentiable at $x = 0$, with $f'(0) = a_1$
3. **Equation of the Tangent Line:** The equation of the tangent line to $f(x)$ at $x = 0$ is: $y = a_0 + a_1 x$

Example 12. *Let* $f(x) = \frac{e^x - 1}{x}$ *with* $e^x = 1 + x + \frac{x^2}{2!} + o(x^2)$*, then*

$$f(x) = \frac{x + \frac{x^2}{2} + x^2\varepsilon(x)}{x} = 1 + \frac{x}{2} + x^2\varepsilon(x)$$

1. We can extend f *by continuity at* $x = 0$ *by posing* $f(0) = 1$*.*
2. The function is differentiable at $x = 0$ *with* $f'(0) = \frac{1}{2}$
3. The equation of the tangent line to f(x) at x = 0 is $y = 1 + \frac{x}{2}$*.*