Course of Mathematics1

Chapter VI : Linear Algebra

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The 1st year Common core In Sciences and Technology

Badji Mokhtar-Annaba University

Academic Year 2024-2025

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6.1 Laws of internal composition

Definition 6.1.1 *Let G a set. An intenal composition on G is an application of $G \times G$ in G. If we write it down*

$$\begin{array}{rcl} G \times G & \rightarrow & G \\ (a, b) & \rightarrow & a * b \end{array}$$

we are talking about the law $*$ and they say that $a * b$ is the compound of a and b for the law $*$.

Example 6.1.1 *On $G = \mathbb{Z}$, the addition defined by*

$$\begin{array}{rcl} \mathbb{Z} \times \mathbb{Z} & \rightarrow & \mathbb{Z} \\ (a, b) & \rightarrow & a + b \end{array}$$

the multplication

$$\begin{array}{rcl} \mathbb{Z}\times\mathbb{Z} & \rightarrow & \mathbb{Z} \\ (a,b) & \rightarrow & a\times b \end{array}$$

Example 6.1.2 *and the subtraction*

$$\begin{array}{rcl} \mathbb{Z}\times\mathbb{Z} & \rightarrow & \mathbb{Z} \\ (a,b) & \rightarrow & a-b \end{array}$$

are internal composition laws

On $G=\mathbb{R}^2$ *the addition*

$$\begin{array}{rcl} \mathbb{R}^2\times\mathbb{R}^2 & \rightarrow & \mathbb{R}^2 \\ ((x_1,y_1),(x_2,y_2)) & \rightarrow & (x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y2) \end{array}$$

is internal law.

Example 6.1.3 *In* $\mathbb{R}^*$ *we define the law* δ *by :*

$$x\delta y = x+y+\ln|xy|$$

then the law δ *is internal on* $\mathbb{R}^*$*,indeed, soit* $x;y\in\mathbb{R}^*$ *, let's show that* $x\delta y\in\mathbb{R}^*$ *, as*

$$\begin{array}{rcl} (x\delta y=0) & \Leftrightarrow & (x+y+\ln|xy|=0) \\ & \Leftrightarrow & (\ln|xy|=-(x+y)) \\ & \Leftrightarrow & \left(|xy|=e^{-(x+y)}\right) \\ & \Leftrightarrow & (x\neq 0 \text{ and } y\neq 0) \end{array}$$

so $x\delta y \in \mathbb{R}^*$ *is an internal law.*

Definition 6.1.2 *Let* $*$ *an internal law on a set* G.

1) The law $*$ *is commutative if*

$$\forall x, y \in G, \qquad x * y = y * x$$

2) The law $*$ *is associative if*

$$\forall x; y; z \in G, \quad (x * y) * z = x * (y * z)$$

3) The law $*$ *admits on* G *a neutral element, noted* e, *if*

$$\exists e \in G, \quad \forall x \in G, \quad x * e = e * x = x$$

if, you can also, the law $*$ *is commutative, just show that*

$$\forall x \in G, \quad x * e = x$$

Example 6.1.4 *In* $\mathbb{R} - \left\{\frac{1}{2}\right\}$ *The internal law* $*$ *is defined by :*

$$x * y = x + y - 2xy$$

the law $*$ *is internal on* $\mathbb{R} - \left\{\frac{1}{2}\right\}$, *indeed, let* $x, y \in \mathbb{R} - \left\{\frac{1}{2}\right\}$, *show that* $x * y \in \mathbb{R} - \left\{\frac{1}{2}\right\}$ *as*

$$\begin{aligned} x * y &= \frac{1}{2} \Leftrightarrow x + y - 2xy = \frac{1}{2} \\ &\Leftrightarrow x(1 - 2y) - \frac{1}{2}(1 - 2y) = 0 \\ &\Leftrightarrow (1 - 2y)\left(x - \frac{1}{2}\right) = 0 \\ &\Leftrightarrow \left(y - \frac{1}{2}\right)\left(x - \frac{1}{2}\right) = 0 \\ &\Leftrightarrow y = \frac{1}{2}, \text{ or } x = \frac{1}{2} \end{aligned}$$

so $x * y \in \mathbb{R} - \left\{\frac{1}{2}\right\}$ *and then* $*$ *is an internal law. Let* x, y; $z \in \mathbb{R} - \left\{\frac{1}{2}\right\}$, *we have*

$$x * y = x + y - 2xy = y + x - yx = y * x$$

then the law $*$ *is commutative.*

$$\begin{aligned} (x * y) * z &= (x + y - 2xy) * z = (x + y - 2xy) + z - 2(x + y - 2xy) z \\ &= x + y + z - 2xy - 2xz - 2yz + 4xyz \\ &= x + (y + z - 2yz) - 2x(y + z - 2yz) \\ &= x + (y * z) - 2x(y * z) = x * (y * z) \end{aligned}$$

then the law $*$ *is associative. Let* $e \in \mathbb{R} - \left\{\frac{1}{2}\right\}$, *such that* $x * e = e * x = x$, *then*

$$x + e - 2xe = e + x - 2ex = x \Leftrightarrow e(1 - 2x) = 0 \Leftrightarrow e = 0$$

then a law accepts as the neutral element the element $e = 0$.

Definition 6.1.3 *Let* $*$ *an internal law on a set* G, *having a neutral element* e *and let* $x \in G$. *It is said that* x *accepts a symmetrical element* x' *by the law* $\star$, *si*

$$x * x' = x' * x = e$$

Example 6.1.5 *On* $\mathbb{R} - \left\{\frac{1}{2}\right\}$, *we define the internal law* $\star$ *by :*

$$x * y = x + y - 2xy$$

the law $*$ *admits like neutral element. Soit* $x \in \mathbb{R} - \left\{\frac{1}{2}\right\}$, *such that* $x \star x' = x' * x = e$, *then*

$$x + x' - 2xx' = x'(1 - 2x) = -x \Leftrightarrow x' = \frac{x}{2x - 1},$$

then, the symmetrical element of x *is*

$$x' = \frac{x}{2x - 1}, \text{ for all } x \in \mathbb{R} - \left\{\frac{1}{2}\right\}$$

show that $x' \in \mathbb{R} - \left\{\frac{1}{2}\right\}$. *Indeed, let* $x, y \in \mathbb{R} - \left\{\frac{1}{2}\right\}$, *show that* $x \star y \in \mathbb{R} - \left\{\frac{1}{2}\right\}$, *we have*

$$x' = \frac{1}{2} \Leftrightarrow 2x - 1 = 2x \Leftrightarrow -1 = 0$$

which is absurd, hence $x' \in \mathbb{R} - \left\{\frac{1}{2}\right\}$.

Definition 6.1.4 *Let* G *a set provided with two internal composition laws, denoted* Δ *and* $\star$. *They say* $*$ *is distributive in relation to* Δ *if*

$$\forall x, y, z \in G, \quad x * (y \Delta z) = (x * y) \, \Delta \, (x * z)$$

6.1.1 Group Structure

Definition 6.1.5 *Let* G *provided with a law of internal composition* $*$. *It is said that* $(G, *)$ *is a group if the law* $*$ *satisfies the following three conditions :*

1) $*$ *is associative.*

2) $*$ *admits a neutral element*

3) Each element of G *allows a symmetrical for* $*$.

If, moreover, the law is commutative, it is said that the group is commutative or abelian (named after the mathematician Abel).

Example 6.1.6 *1)* $(\mathbb{Z}, +)$ *is a commutative group.*

2) $(\mathbb{R}, \times)$ *is not a group because 0 does not allow symmetrical elements.*

3) $(\mathbb{R}^*, \times)$ *is a commutative group.*

Definition 6.1.6 *Let* $(G, *)$ *a group. A part* $H \subset G$ *(non-empty) is a subgroup of* G *if the restriction of the operation* $*$ *to* H *gives it the group structure.*

Proposition 6.1.1 *Let* H *is a non-empty part of the group* G. *Then,* H *is subgroup de* G *if and only of,*

1) For all $x, y \in H$, *we have* $x * y \in H$,

2) For all $x \in H$, *we have* $x' \in H$, *avec* $x' \in$ *is the symmetrical of* x.

Example 6.1.7 $\left(\mathbb{R}_+^*, \times\right)$ *is a subgroup of* $\left(\mathbb{R}_+^*, \times\right)$. *Indeed :*

1) Si $x, y \in \mathbb{R}_+^*$, *alors* $x \times y \in \mathbb{R}_+^*$,

2) Si $x \in \mathbb{R}_+^*$ *alors* $x' = \frac{1}{x}$ *symmetrical element of* x *and* $x' = \frac{1}{x} \in \mathbb{R}_+^*$.

Example 6.1.8 *We pose* $2\mathbb{Z} = \{2z : z \in \mathbb{Z}\}$, $\{2\mathbb{Z}, +\}$ *is a subgroup of* $\mathbb{Z}$. *Indeed :*

1) If $x, y \in 2\mathbb{Z}$, *it exists* $x_1 \in \mathbb{Z}$ *such that* $x = 2x_1$ *and* $y = 2y_1$, *then*

$$x + y = 2x_1 + 2y_1 = 2(x_1 + y_1) \in 2\mathbb{Z}.$$

2) If $x \in 2\mathbb{Z}$, *it exists* $x_1 \in \mathbb{Z}$ *such that* $x = 2x_1$ *then*

$$-x = -2x_1 = 2(-x) \in 2\mathbb{Z}.$$

6.1.2 Ring structure

Definition 6.1.7 *Let* A *a set provided with two internal composition laws, denoted* Δ *and* $\star$. $(A, \Delta, *)$ *is said to be a ring if the following conditions are satisfies :*

1) (A, Δ) *is a commutative group.*

2) The law $*$ *is associative.*

3) The law $*$ *is distributive in relation to the law* Δ.

If, moreover, the law $*$ *is commutative, it said that the law* $(A, \Delta, *)$ *is commutative.*

If the law $*$ *allows a neutral element, it is said that the ring* $(A, \Delta, *)$ *is unitary.*

Example 6.1.9 $(\mathbb{Z}, +, .)$ *is a ring commutative and unitary.*

6.1.3 structure of a body

Definition 6.1.8 *Let* $\mathbb{k}$ *a set provided with two internal composition laws, denoted* Δ *and* $\star$. $(A, \Delta, *)$ *is said to be a body if the following conditions are satisfies :*

1) $(\mathbb{k}, \Delta, *)$ *is a ring.*

2) $(\mathbb{k} - \{e\}, \Delta)$ *is a group, hence* e *is the neutral element of* $*$.

Example 6.1.10 $(\mathbb{R}, +, .)$ *is a commutative body.*

6.2 Vector Espace

6.2.1 Definitions and elementary properties

Let $\mathbb{k}$ be a commutative body (usually it's $\mathbb{R}$ or $\mathbb{C}$) and let E be a non-empty assembly provided with an internal operation denoted by $(+)$

$$(+) : E \times E \rightarrow E$$
$$(x, y) \rightarrow (x + y)$$

and an external operation noted $(.)$

$$(.) : \mathbb{k} \times E \rightarrow E$$
$$(\lambda, y) \rightarrow (\lambda.y)$$

Definition 6.2.1 *A vector space on the body $\mathbb{k}$ or a $\mathbb{k}-$vector space is a triplet $(E, +, .)$ such that :*

1) $(E, +)$ is a commutative group.

2) $\forall\lambda \in \mathbb{k}, \forall x, y \in E,\ \lambda. (x + y) = \lambda.x + \lambda.y.$

3) $\forall\lambda, \mu \in \mathbb{k}, \forall x \in E,\ (\lambda + \mu) .x = \lambda.x + \mu.x.$

4) $\forall\lambda, \mu \in \mathbb{k}, \forall x \in E,\ (\lambda.\mu) .x = \lambda (.\mu.x)$

5) $\forall x \in E, 1_{\mathbb{k}}.x = x$

The elements of the vector space are called vectors and those of $\mathbb{k}$ are called scalars.

Example 6.2.1 *1) $(\mathbb{R}, +, .)$ is a $\mathbb{R}-$vector space,*

2) $(\mathbb{C}, +, .)$ is a $\mathbb{C}-$vector space,

3) $(\mathbb{C}, +, .)$ is a $\mathbb{R}-$vector space,

4) If we consider $\mathbb{R}^n$ providded with the following two operations

$$(+) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$
$$((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)) \rightarrow (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$$

$$(.) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$
$$(\lambda (y_1, y_2, ..., y_n)) \rightarrow (\lambda y_1, \lambda y_2, ..., \lambda y_n)$$

It can easily shown that $(\mathbb{R}^n, +, .)$ *is a* $\mathbb{R}-$*vector space.*

Proposition 6.2.1 *If E is a $\mathbb{k}-$vector space, then we have the following properies :*

1) $\forall x \in E,\ 0_{\mathbb{k}}.x = 0_E.$

2) $\forall x \in E,\ (-1_{\mathbb{k}}).x = x$

3) $\forall \lambda \in \mathbb{k},\ \lambda.0_E = 0_E$

4) $\forall \lambda \in \mathbb{k},\ \forall x, y \in E, \lambda (x - y) = \lambda.x - \lambda.y$

5) $\forall \lambda \in \mathbb{k},\ \forall x \in E,\ \lambda.x = 0_E \Leftrightarrow \lambda = 0_{\mathbb{k}}$ *or* $x = 0_E.$

Definition 6.2.2 *Let $(E, +, .)$ be $\mathbb{k}-$vector space and let F be non-empty sub set of F. it is said that F is vector subspace if $(F, +, .)$ is also a $\mathbb{k}-$vector space.*

Remark 6.2.1 *1) When $(F, +, .)$ is $\mathbb{k}-$vector space of $(E, +, .)$ then $0_E \in F$.*

2) If $0_E \notin F$, then $(F, +, .)$ can't be a $\mathbb{k}-$vector space of $(E, +, .)$.

Theoreme 6.2.1 *Let $(E, +, .)$ be $\mathbb{k}-$vector space and $F \subset E$, F non empty we have the following equivalence :*

1) F is a vector subspace of E.

2) F is stable by addition and by multiplication, i.e :

$$\forall \lambda \in \mathbb{k},\ \forall x, y \in F,\ \lambda.x \in F \text{ and } x + y \in F$$

3) $\forall \lambda, \mu \in \mathbb{k}, \forall x, y \in F, \lambda x + \mu y \in F$, *so*

$$F \text{ is a vector subspace} \quad \Leftrightarrow \begin{cases} F \neq \varnothing \\ \forall \lambda, \mu \in \mathbb{k}, \forall x, y \in F, \lambda.x + \mu.y \in F \end{cases}$$

Example 6.2.2 *We pose $F = \{(x, y) \in \mathbb{R}^2 : x - y = 0\} \subset \mathbb{R}^2$, then F is a vector subspace, indeed,*

1) $0_{\mathbb{R}^2} = (0, 0) \in F$, *because* $0 - 0 = 0$

2) $\forall \lambda, \mu \in \mathbb{R}, \forall (x, y), (x', y') \in F$, *then* $x - y = 0$, *and* $x' - y' = 0$, *so*

$$\lambda (x - y) + \mu (x' - y') = (\lambda x + \mu x') - (\lambda y + \mu y') = 0,$$

i.e, $\lambda (x, y) + \mu (x', y') \in F$, *so F is vector subspace of* $\mathbb{R}^2$.

Proposition 6.2.2 *Tthe intersection of a non-empty family of vector subspaces is a vector subspace.*

Remark 6.2.2 *Reuniting two vector subspaces is not necessarily a vector subspace.*

Example 6.2.3 *Let* $F_1 = \{(x, y) \in \mathbb{R}^2 : x = 0\}$ *and* $F_2 = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ *two vector subspaces in* $\mathbb{R}^2$, $F_1 \cup F_2$ *is not a vector subspace, because*

$$u_1 = (1, 0) \in F_1, u_1 = (0, 1) \in F_2 \text{ and } u_1 + u_2 = (1, 1) \notin F_1 \cup F_2$$

6.2.2 Sum of two vector subspaces

Definition 6.2.3 *Let* E_1, E_2 *two vector subspaces of* $\mathbb{k}-$*vector space* E, *it said Sum of two vector subspaces,* E_1 *and* E_2, *which we note* $E_1 + E_2$ *the following set :*

$$E_1 + E_2 = \{x \in E : \exists x_1 \in E_1, \exists x_2 \in E_2 \text{ such that } x = x_1 + x_2\}$$

Example 6.2.4 *Let's* $E_1 = \{(x, y) \in \mathbb{R}^2 : x = 0\}$ *and* $E_2 = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ *two vector subspaces in* $\mathbb{R}^2$, *if* $(x, y) \in \mathbb{R}^2$, *then*

$$(x, y) = \underset{\in E_1}{(0, y)} + \underset{\in E_2}{(x, 0)}$$

So $(x, y) \in E_1 + E_2$, *then* $E_1 + E_2 = \mathbb{R}^2$.

Proposition 6.2.3 *The sum of two vector subspaces* E_1 *and* E_2 *(of the same* $\mathbb{k}-$*vector space) is a vector subspace of* E *container* $E_1 \cup E_2$, i, e,

$$E_1 \cup E_2 \subset E_1 + E_2$$

6.2.3 Direct sum of two vector subspaces

Definition 6.2.4 *Let* E_1, E_2 *two vector subspaces of the same* $\mathbb{k}-$*vector space* E. *It will be said that the sum* $E_1 \oplus E_2$ *of two vector subspaces , is direct if* $E_1 \cap E_2 = \{0\}$ *. We write* $E_1 \oplus E_2$.

Proposition 6.2.4 *Let E_1, E_2 two vector subspaces of the same $\mathbb{k}-$vector space E. The sum $E_1 + E_2$ is direct if $\forall x \in E_1 + E_2$, there is a single vector $x_1 \in E_1$, a single vector $x_2 \in E_2$ such that $x = x_1 + x_2$*

Example 6.2.5 *Let's $F_1 = \{(x, y, z) \in \mathbb{R}^3 : x = 0\}$ and $F_2 = \{(x, y, z) \in \mathbb{R}^3 : y = z = 0\}$ two vector subspaces in $\mathbb{R}^3$.*

1) Let $(x, y, z) \in \mathbb{R}^3$, then

$$(x, y, z) = \underset{\in F_1}{(0, y, z)} + \underset{\in F_2}{(x, 0, 0)}$$

then $(x, y, z) \in F_1 + F_2$, hence $F_1 + F_2 = \mathbb{R}^3$.

2) Let $(x, y, z) \in F_1 \cap F_2$, then $(x, y, z) \in F_1$ and $(x, y, z) \in F_2$, it means that $x = 0$ and $y = z = 0$, then $(x, y, z) = 0_{\mathbb{R}^3}$, i,e, $F_1 \cap F_2 = \{0\}$.

Finally, we conclude that $\mathbb{R}^3 = F_1 \oplus F_2$.

6.2.4 Generating families, free families and bases

Hereinafter, the vector space $(E, +, .)$ will be designated by E.

Definition 6.2.5 *Let E be a vector space and $e_1, e_2, ..., e_n$ elements of E.*

1) They say that $\{e_1, e_2, ..., e_n\}$ are free or linearly independent, if for all $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{k}$:

$$\alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n = 0_E \Rightarrow \alpha_1 = \alpha_2 = ... = \alpha_n = 0_{\mathbb{k}}$$

if they are not, they are said to be related.

2) They say that $\{e_1, e_2, ..., e_n\}$ is a generates family E, or that E is generated by $\{e_1, e_2, ..., e_n\}$ if

$$\forall x \in E, \exists \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{k}, \quad x = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n$$

3) If $\{e_1, e_2, ..., e_n\}$ is a free and generates family of E, so $\{e_1, e_2, ..., e_n\}$ is called a base of E.

Example 6.2.6 *On $\mathbb{R}^2$, we pose $u_1 = (1, 0)$, $u_2 = (1, -1)$, then $\{u_1, u_2\}$ is a base of $\mathbb{R}^2$. Indeed* *i)* $\{u_1, u_2\}$ *is free.* $\forall \alpha_1, \alpha_2 \in \mathbb{R}$,

$$\begin{aligned}(\alpha_1 u_1 + \alpha_2 u_2 = 0) &\Rightarrow \alpha_1 (1, 0) + \alpha_2 (1, -1) = (0, , 0) \\ &\Rightarrow (\alpha_1 + \alpha_2, -\alpha_2) = (0, , 0) \\ &\Rightarrow \alpha_1 = \alpha_2 = 0\end{aligned}$$

ii) $\{u_1, u_2\}$ *is generating.* $\forall (x, y) \in \mathbb{R}^2$,

$$(x, y) = \alpha_1 u_1 + \alpha_2 u_2 = (\alpha_1 + \alpha_2, -\alpha_2) \Rightarrow \alpha_2 = -y \in \mathbb{R} \text{ and } \alpha_1 = x + y \in \mathbb{R},$$

then it existe $\alpha_1, \alpha_2 \in \mathbb{R}$

Remark 6.2.3 *In a vecctor space E, all non-zero vector is free.*

Example 6.2.7 *in all the polynomials of degree less than or equal to 2 with real coefficients and with an indeterminate x*

$$\mathbb{R}_2 [x] = \left\{P(x) = a + bx + cx^2 : a, b, c \in \mathbb{R}\right\}$$

then $\{p_1(x) = 1, p_2(x) = x, p_3(x) = x^2\}$ *is a base familly. Indeed*
i) Let $\alpha, \beta, \gamma \in \mathbb{R}$, *then*

$$\forall x \in \mathbb{R} : \alpha p_1(x) + \beta p_2(x) + \gamma p_3(x) = 0 \Leftrightarrow \forall x \in \mathbb{R} : \alpha + \beta x + \gamma x^2 = 0$$

what gives $\alpha = \beta = \gamma = 0$, *then then* $\{1, x, x^2\}$ *is a free family.*
ii) Let $P \in \mathbb{R}_2 [x]$, *then it exists* $a, b, c \in \mathbb{R}$, *such that*

$$\forall x \in \mathbb{R} : P(x) = a + bx + cx^2 = ap_1(x) + bp_2(x) + cp_3(x)$$

i,e,

$$P = ap_1 + bp_2 + cp_3$$

then $\{1, x, x^2\}$ *is generating.*

Proposition 6.2.5 *If* $\{e_1, e_2, ..., e_n\}$ *and* $\{u_1, u_2, ..., u_m\}$ *are two bases of the vector space E, then* $n = m$.

Remark 6.2.4 *If a vector space E admits a base then all the bases of E have the same number of elements (or same cardinal), this number does not depend on the base but it depends only on the space E.*

Definition 6.2.6 *Let E be a $\mathbb{k}-$vectoriel space of base $B = \{e_1, e_2, ..., e_n\}$, The dimension of E, denoted $\dim E$, is the number defined by $\dim(E) = Card(B)$ where $Card(B)$ is the cardinal of B.*

Example 6.2.8 *We pose $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$, then $\{e_1, e_2, e_3\}$ is a base of $\mathbb{R}^3$, so*

$$\dim\left(\mathbb{R}^3\right) = Card\left(\{e_1, e_2, e_3\}\right) = 3$$

Example 6.2.9 *On $\mathbb{R}_2[x]$, the family $\{1, x, x^2\}$ is a base of $\mathbb{R}_2[x]$*

$$\dim \mathbb{R}_2[x] = Card\left\{1, x, x^2\right\} = 3$$

Theoreme 6.2.2 *Let E a vector space of dimension n, then*

1) If $\{e_1, e_2, ..., e_n\}$ is a base of $E \Leftrightarrow \{e_1, e_2, ..., e_n\}$ is genetary $\Leftrightarrow \{e_1, e_2, ..., e_n\}$ is free.

2) If $\{e_1, e_2, ..., e_p\}$ are p vectors in E, with $p \succ n$, then $\{e_1, e_2, ..., e_p\}$ cannot be free, moreever si $\{e_1, e_2, ..., e_p\}$ is generaty, then there are n parmis vectors $\{e_1, e_2, ..., e_p\}$ which form a basis E.

3) If $\{e_1, e_2, ..., e_p\}$ are p vectors in E, with $p < n$, then $\{e_1, e_2, ..., e_p\}$ cannot be generaty, then it exist $(n-p)$ vectors $\{e_{p+1}, e_{p+2}, ..., e_n\}$ on E suc that $\{e_1, e_2, ..., e_{p+1}, .., e_n\}$ is a basis for E.

4) If F be a vector subspace of E then $\dim F \leq n$, and more $\dim F = n \Leftrightarrow F = E$.

6.3 Linear application

6.3.1 Definitions

Definition 6.3.1 *Let's E and F two $\mathbb{k}-$vector spaces. An application f of E on F is linear application if satisfies the following two conditions :*

$$\begin{array}{ll} \forall x, y \in E, & f(x+y) = f(x) + f(y) \\ \forall x \in E, \forall \lambda \in \mathbb{k}, & f(\lambda x) = \lambda f(x) \end{array}$$

where in an equivalent manner

$$\forall x, y \in E, \quad \forall \lambda \in \mathbb{k}, \quad f(\lambda x + y) = \lambda f(x) + f(y)$$

Remark 6.3.1 *The set's of the linear application of E on F denoted $\mathcal{L}(E, F)$.*

Example 6.3.1 *The application f defined by*

$$\begin{array}{l} f : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \\ (x, y, z) \rightarrow f(x, y, z) = (2x + y, y - z) \end{array}$$

is a linear application. Indeed, let's (x, y, z) ,$(x', y', z') \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} f[(x, y, z) + (x', y', z')] &= f(x + x', y + y', z + z') \\ &= (2(x + x') + (y + y'), (y + y') - (z + z')) \\ &= (2x + 2x' + y + y', y + y' - z - z') \\ &= ((2x + y) + (2x' + y'), (y - z) + (y' - z')) \\ &= (2x + y, y - z) + (2x' + y', y' - z') \\ &= f(x, y, z) + f(x', y', z') \end{aligned}$$

and

$$\begin{aligned} f\left[\lambda\left(x,y,z\right)\right] &= f(\lambda x, \lambda y, \lambda z) = (2\lambda x + \lambda y, \lambda y - \lambda z) = \left(\lambda\left(2x+y\right), \lambda\left(y-z\right)\right) \\ &= \lambda\left(2x+y, y-z\right) \\ &= \lambda f(x,y,z) \end{aligned}$$

Remark 6.3.2 *All the applications aren't linear applications.*

Definition 6.3.2 *Let's E and F are two $\mathbb{k}-$vector spaces, and let $f \in \mathcal{L}\left(E,F\right)$. They say that*

1) f is an isomorphism of E on F, if f is bijective.

2) f is an endomorphism, if $(E,+,.) = (F,+,.)$.

3) f is an autorphism, if f is endorphism and isomorphism.

Example 6.3.2 *The application f defined by*

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R} \\ x &\to f(x) = -2x \end{aligned}$$

is an automorphisme, Indeed, let $x, y, \lambda \in \mathbb{R}$, we have

$$f(\lambda x + y) = -2\left(\lambda x + y\right) = \lambda\left(-2x\right) + \left(-2y\right) = \lambda f(x) + f(y)$$

and the application f is bijective, where

$$\begin{aligned} f^{-1} &: \mathbb{R} \to \mathbb{R} \\ x &\to f^{-1}(x) = \frac{-1}{2}x \end{aligned}$$

Notation.

The null application, denoted $0_{\mathcal{L}(E,F)}$ is given by :

$$f : E \to F, \quad x \to f(x) = 0_F$$

the identity application, noted id_E is given by :

$$id_E : E \to F, \quad x \to id_E(x) = x$$

Proposition 6.3.1 *Let f is a linear application of E on F, we have*

1) $f(0_E) = 0_F$

2) $\forall x \in E : f(-x) = -f(x)$

Proof Let $x \in E$, we have

$$\begin{aligned} 1) f(0_E) &= f(0_{\mathbb{k}}.0_E) = 0_{\mathbb{k}}.f(0_E) = 0_F, \\ 2) f(-x) &= f((-1).x) = (-1)f(x) = -f(x) \end{aligned}$$

■

6.3.2 Kernel, image, and rank of a linear application

Definition 6.3.3 *Let f be a linear application of E on F.*

1) The set $f(E)$ is called the image of the linear application f and is denoted $\operatorname{Im} f$ *i.e*

$$\operatorname{Im} f = \{f(x) : x \in E\}$$

2) The set $f^{-1}(\{0\})$ is calledis the kernel of the linear application and is denoted $\ker f$ *i.e*

$$\ker f = \{x \in E, \quad f(x) = 0_F\}$$

Example 6.3.3 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a linear application defined by*

$$(x, y) \to f(x, y) = x - y$$

The kernel of the linear application f,

$$\begin{aligned} \ker f &= \left\{(x, y) \in \mathbb{R}^2 : x - y = 0\right\} \\ &= \left\{(x, y) \in \mathbb{R}^2 : x = y\right\} \\ &= \{x(1, 1), \ x \in \mathbb{R}\} \end{aligned}$$

then the $\ker f$ *is a vector subspace generated by* $e = (1, 1)$ *then it's dimension one, and its base is* $\{e\}$.

The image of the linear application f,

$$\begin{aligned} \operatorname{Im} f &= \left\{ f(x, y) : (x, y) \in \mathbb{R}^2 \right\} \\ &= \left\{ x - y, (x, y) \in \mathbb{R}^2 \right\} = \mathbb{R} \end{aligned}$$

Proposition 6.3.2 *Let* f *be a linear application of* E *on* F.

1) $\operatorname{Im} f$ *is a vector subspace of* F.

2) $\ker f$ *is a vector subspace of* E.

Definition 6.3.4 *Let* f *be a linear application of* E *in* F, *if* $\dim \operatorname{Im} f = n < +\infty$, *then* n *its said the rank of* f *and and the note* $rg(f)$.

Proposition 6.3.3 *Let* f *be a linear application of* E *in* F. *we have the following equivalences :*

(i) f *is surjective* $\Leftrightarrow \operatorname{Im} f = F$

(ii) f *is injective* $\Leftrightarrow \ker f = \{0_E\}$.

Example 6.3.4 *Let* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *a linear application defined by*

$$(x, y) \to f(x, y) = (y, x)$$

we have

$$\begin{aligned} \operatorname{Im} f &= \left\{ f(x, y) : (x, y) \in \mathbb{R}^2 \right\} = \left\{ (y, x) : (x, y) \in \mathbb{R}^2 \right\} \\ &= \left\{ y(1, 0) + x\,(0, 1) : (x, y) \in \mathbb{R}^2 \right\}, \end{aligned}$$

and

$$\ker f = \left\{ (x, y) \in \mathbb{R}^2 : (y, x) = 0_{\mathbb{R}^2} \right\} = \{(0, 0)\}$$

then $\operatorname{Im} f = \mathbb{R}^2$ *and* $\ker f = \{0_{\mathbb{R}^2}\}$, *then* f *is bijective.*

6.3.3 linear application to finite dimension spaces

Proposition 6.3.4 *Let E and F two $\mathbb{k}-$vector spaces and f and two linear applications of E in F. If E is finite dimension n and $\{e_1, e_2, ..., e_n\}$ is a base of E, then*

$$\forall k \in \{1, 2, ..., n\} : f(e_k) = g(e_k) \Leftrightarrow \forall x \in E : f(x) = g(x)$$

Proof the implication ($\Leftarrow$) is obvious.

For ($\Rightarrow$) we have E is generated by $\{e_1, e_2, ..., e_n\}$, donc

$$\forall x \in E, \exists \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{k} : x = \lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_n e_n,$$

as f and g are linears, then

$$\begin{aligned} f(x) &= f(\lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_n e_n) = \lambda f(e_1) + \lambda_2 f(e_2) + ... + \lambda_n f(e_n) \\ g(x) &= g(\lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_n e_n) = \lambda g(e_1) + \lambda_2 g(e_2) + ... + \lambda_n g(e_n) \end{aligned}$$

so if we suppose that $\forall k \in \{1, 2, ..., n\} : f(e_k) = g(e_k)$ then we deduce

$$\forall x \in E, f(x) = g(x)$$

■

Example 6.3.5 *Let f be an application of $\mathbb{R}^2$ in $\mathbb{R}$ such that*

$$f(1, 0) = -1 \text{ and } f(0, 1) = 4,$$

then $\forall (x, y) \in \mathbb{R}^2$, we have

$$\begin{aligned} f(x, y) &= f[x(1, 0) + y(0, 1)] = xf(1, 0) + y(0, 1) \\ &= -x + 4y \end{aligned}$$

Proposition 6.3.5 *Let f be a linear application of E in F with dimension of E is finite, we have*

$$\dim E = \dim \ker(f) + \dim \operatorname{Im}(f)$$

Example 6.3.6 *Let f be a linear application of $\mathbb{R}^2$ in $\mathbb{R}$ defined by*

$$f(x,y) = -x + 5y$$

we have

$$\begin{aligned} \ker(f) &= \left\{\forall\,(x,y) \in \mathbb{R}^2 : f(x,y) = 0\right\} = \left\{(x,y) \in \mathbb{R}^2 : x = 5y\right\} \\ &= \left\{y\,(5,1) : y \in \mathbb{R}\right\}, \end{aligned}$$

then $\dim\ker(f) = 1$ *as* $\dim\mathbb{R}^2 = 1,$ *then*

$$\dim\operatorname{Im}(f) = \dim\mathbb{R}^2 - \dim\ker(f) = 1$$

Proposition 6.3.6 *Let f be a linear application of E in F with* $\dim E = \dim F = n$*. The following equivalences are then obtained*

$$\begin{aligned} f \text{ is isomrphism} &\Leftrightarrow f \text{ is surjective} \Leftrightarrow \dim\operatorname{Im}(f) = \dim F \\ &\Leftrightarrow f \text{ is injective} \Leftrightarrow \operatorname{Im}(f) = F \\ &\Leftrightarrow \dim\ker(f) = 0 \Leftrightarrow \ker(f) = \{0\} \end{aligned}$$

Remark 6.3.3 *Of this proposition, we deduce that f is isomorphism of E in F with* $\dim E$ *finite then necessarily* $\dim E = \dim F$ *in other words, if* $\dim E \neq \dim F$ *then f cannot be an isomorphism.*

Example 6.3.7 *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$f(x,y) = (2x - y, x)$$

we have

$$\begin{aligned} \ker(f) &= \left\{(x,y) \in \mathbb{R}^2 : f(x,y) = 0\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 : 2x - y = x = 0\right\} \\ &= \{(0,0)\} \end{aligned}$$

as $\dim f = 2$ *and* $\ker(f) = \{0_{\mathbb{R}^2}\}$, *then* f *is an isomorphism.*

Exercise 1 *We define in* $G =]-1,1[$ *the internal law* $*$ *as follows*

$$\forall (x,y) \in G \times G : x * y = \frac{x+y}{1+xy}$$

Show that $(G,*)$ *is a commutatif group.*

Solution

the law $*$ is internal in $]-1,1[$. Indeed, let's $x, y \in]-1,1[$, let's show that $x * y \in]-1,1[$. We have

$$\begin{aligned} x * y &\in]-1,1[\Leftrightarrow |x*y| < 1 \Leftrightarrow \frac{|x+y|}{|1+xy|} < 1 \\ &\Leftrightarrow |x+y| < |1+xy| \Leftrightarrow (x+y)^2 < (1+xy)^2 \\ &\Leftrightarrow x^2\left(1-y^2\right) - \left(1-y^2\right) < 0 \\ &\Leftrightarrow \left(1-x^2\right)\left(1-y^2\right) \succ 0 \end{aligned}$$

as $x, y \in]-1,1[$, then $(1-x^2)(1-y^2) \succ 0$, hence $x * y \in]-1,1[$ and then $*$ is an internaal law.

The law $*$ is commutative : for all $(x,y,z) \in G^3$

$$x * y = \frac{x+y}{1+xy} = \frac{y+x}{1+yx} = y * x$$

The law $*$ is associative : for all $(x,y,z) \in G^3$

$$x * (y * z) = x * \left(\frac{y+z}{1+yz}\right) = \frac{x + \frac{y+z}{1+yz}}{1 + x\frac{y+z}{1+yz}} = \frac{x+y+z+xyz}{1+yz+xy+xz}$$

and a similar calculation gives the same result for $(x * y) * z$.

The law $*$ admits a neutral element, because for all $x \in]-1,1[$

$$\begin{aligned} (x * e = x) &\Leftrightarrow \frac{x+e}{1+xe} = x \Leftrightarrow x + e = x(1+xe) \\ &\Leftrightarrow x^2 e = e \Leftrightarrow e = 0, \text{ because } x^2 \neq 1 \end{aligned}$$

then $e = 0$ is the neutral element for the law $*$.

The element of G admits an inverse in G. Let $x \in G$, then

$$(x * x' = e) \Leftrightarrow \frac{x + x'}{1 + xx'} = 0 \Leftrightarrow x + x' = 0 \Leftrightarrow x' = -x \in]-1, 1[\,,$$

donc the inverse of x is $-x$, and then $(G, *)$ is a ccommutatif groupe.

Exercise 2 *Let* $E = \{(x, y, z) \in \mathbb{R}^3 : x + y - 2z = 2x - y - z = 0\}$ *a sub-set of* $\mathbb{R}^3$.

(1) Show that E *is a vector subspace of* $\mathbb{R}^3$.

(2) Determine a family generates of E *and extract a basis from it ?*

(3) Let $F = \{(x, y, z) \in \mathbb{R}^3 : x + y - z = 0\}$ *a vector subspace of* $\mathbb{R}^3$.

(i) Determine a generates family of F?

(ii) Have we $E \oplus F = \mathbb{R}^3$?

Solution

(1) Let $u = (x, y, z) \in E$, then

$$\begin{cases} x + y - 2z = 0 \\ 2x - y - z = 0 \end{cases} \Leftrightarrow \begin{cases} x + y - 2z = 0 \\ 3x - 3z = 0 \end{cases} \Leftrightarrow \begin{cases} x = y \\ x = z \end{cases}$$
$$\Leftrightarrow \quad x = y = z$$

then

$$E = \left\{(x, y, z) \in \mathbb{R}^3 : x = y = z\right\}$$

. $0_{\mathbb{R}^3} \in E$, because $0 = 0 = 0$, so $E \neq \varnothing$.

. Let's $u = (x, y, z) \in E$, $v = (x', y', z') \in E$, so we have $x = y = z$ and $x' = y' = z'$. Let $\lambda, \mu \in \mathbb{R}$, then

$$\lambda u + \mu v = \lambda(x, y, z) + \mu(x', y', z') = \Big(\underbrace{\lambda x + \mu x'}_{x''}, \underbrace{\lambda y + \mu y'}_{y''}, \underbrace{\lambda z + \mu z'}_{z''}\Big)$$

as $\lambda x + \mu x' = \lambda y + \mu y' = \lambda z + \mu z'$, then $x'' = y'' = z''$, which shows that $\lambda u + \mu v \in E$.

(2) We have

$$E = \{(x, x, x) : x \in \mathbb{R}\} = \left\{x \underbrace{(1, 1, 1)}_{u_1} : x \in \mathbb{R}\right\}$$

then $\{u_1\}$ is a generates family of E, so $\{u_1\}$ is a base of E.
$(3 - i)$ Let $(x, y, z) \in F$, then $z = x + y$, so

$$F = \{(x, y, x + y) : x, y \in \mathbb{R}\} = \left\{x \underbrace{(1, 0, 1)}_{u_2} + y \underbrace{(0, 1, 1)}_{u_3} : x, y \in \mathbb{R}\right\}$$

then $\{u_2, u_3\}$ is a generates family of F. Show that $\{u_2, u_3\}$ is free. Let's $\lambda_2, \lambda_3 \in \mathbb{R}$,

$$\lambda_2 u_{2+} \lambda_3 u_3 = 0_{\mathbb{R}^3} \Rightarrow (\lambda_2, \lambda_3, \lambda_2 + \lambda_3) = (0, 0, 0) \Rightarrow \lambda_2 = \lambda_3 = 0,$$

so $\{u_2, u_3\}$ is a base of F.
$(3 - ii)$ As $\{u_1\}$ is a base of E, $\{u_2, u_3\}$ is a basis of F, then if $\{u_1, u_2, u_3\}$ is a basis of $\mathbb{R}^3$, we have $E \oplus F = \mathbb{R}^3$, since $Card\{u_1, u_2, u_3\} = \dim \mathbb{R}^3 = 3$, vyou just have to prove $\{u_1, u_2, u_3\}$ is free. Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$,

$$\begin{aligned} \lambda_1 u_1 + \lambda_2 u_{2+} \lambda_3 u_3 &= 0_{\mathbb{R}^3} \Rightarrow (\lambda_2 + \lambda_1, \lambda_3 + \lambda_1, \lambda_2 + \lambda_3) = (0, 0, 0) \\ &\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0 \end{aligned}$$

then $\{u_1, u_2, u_3\}$ is a basis of $\mathbb{R}^3$, so $E \oplus F = \mathbb{R}^3$.

Exercise 3 *We consider the application $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by :*

$$f(x, y) = (x - y, -3x + 3y)$$

(1) Show that f is a linear application.
(2) Give a basis of its core and a basis of his image.
(3) Determine $f \circ f$.

Solution

(1) Let's $u=(x,y)\in\mathbb{R}^2, v=(x',y')\in\mathbb{R}^2$ and $\alpha,\beta\in\mathbb{R}$, we have

$$\begin{aligned} f(\alpha u+\beta v) &= f\left(\alpha x+\beta x', \alpha y+\beta y'\right) \\ &= \left(\alpha x+\beta x'-\alpha y-\beta y', -3\alpha x-3\beta x'+3\alpha y+3\beta y'\right) \\ &= \left(\left(\alpha x-\alpha y\right)+\left(\beta x'-\beta y'\right), \left(-3\alpha x+3\alpha y\right)+\left(-3\beta x'+3\beta y'\right)\right) \\ &= \alpha\left(x-y,-3x+3y\right)+\beta\left(x'-y',-3x'+3y'\right)=\alpha f(u)+\beta f(v) \end{aligned}$$

which shows that f is linear.
(2) We have

$$\begin{aligned} \ker(f) &= \left\{(x,y)\in\mathbb{R}^3 : f(x,y)=0\right\} \\ &= \left\{(x,y)\in\mathbb{R}^3 : x-y=-3x+3y=0\right\} \\ &= \left\{(x,y)\in\mathbb{R}^3 : x=y\right\} \\ &= \left\{x(1,1) : x\in\mathbb{R}\right\}, \end{aligned}$$

so $\{u_1\}$ is a basis of $\ker(f)$.

$$\begin{aligned} \operatorname{Im}(f) &= \left\{f\left(x,y\right) : (x,y)\in\mathbb{R}^3\right\} \\ &= \left\{\left(x-y,-3x+3y\right) : (x,y)\in\mathbb{R}^3\right\} \\ &= \left\{\underbrace{\left(x-y\right)}_{\lambda}\left(1,-3\right) : (x,y)\in\mathbb{R}^3\right\} \\ &= \left\{\lambda\left(1,-3\right) : \lambda\in\mathbb{R}\right\} = \left\{\lambda\underbrace{(1,-3)}_{u_2} : \lambda\in\mathbb{R}\right\} \end{aligned}$$

then $\{u_2\}$ is a basis of $\operatorname{Im}(f)$.
(3) Let $(x,y)\in\mathbb{R}^2$, then

$$\begin{aligned} \left(f\circ f\right)(x,y) &= f\left(f(x)\right)=f\left(x-y,-3x+3y\right) \\ &= \left(\left(x-y\right)-\left(-3x+3y\right), -3\left(x-y\right)+3\left(-3x+3y\right)\right) \\ &= \left(x-y+3x-3y, -3x+3y-9x+9y\right) \\ &= \left(4x-4y, -12+12y\right) \end{aligned}$$

Exercise 4 *Let* $f:\mathbb{R}^3\rightarrow\mathbb{R}^3$ *defined by :*

$$f(x,y,z)=(-2x+y+z,x-2y+z,x+y-2z)$$

(1) Show that f *is a linear application.*
(2) Give a basis of $\ker(f)$ *and deduce* $\dim(\mathrm{Im}(f))$
(3) Give a basis of $\mathrm{Im}(f)$*.*

Solution

(1) Let's $u=(x,y,z)\in\mathbb{R}^3, v=(x',y',z')\in\mathbb{R}^3$, and $\alpha,\beta\in\mathbb{R}$, we have

$$\begin{aligned}
f(\alpha u+\beta v) &= (\alpha x+\beta x',\alpha y+\beta y',\alpha z+\beta z') \\
&= (-2\alpha x-2\beta x'+\alpha y+\beta y'+\alpha z+\alpha z',\alpha x+\beta x'-2\alpha y-2\beta y'+\alpha z+\beta z',\alpha x+\beta x \\
&= ((-2\alpha x+\alpha y+\alpha z)+(-2\beta x'+\beta y'+\beta z'),(\alpha x-2\alpha y+\alpha z)+(\beta x'-2\beta y'+\beta z') \\
&= \alpha(2x+y+z,x-2y+z,x+y-2z)+\beta(-2x'+y'+z',x'-2y'+z',x'+y'-2 \\
&= \alpha f(u)+\beta f(v)
\end{aligned}$$

then f is linear.
(2) We have

$$\begin{aligned}
\ker(f) &= \left\{(x,y,z)\in\mathbb{R}^3: f(x,y,z)=0\right\} \\
&= \left\{(x,y,z)\in\mathbb{R}^3: -2x+y+z=x-2y+z=x+y-2z=0\right\}
\end{aligned}$$

Then

$$\begin{aligned}
(x,y,z) &\in \ker(f)\Leftrightarrow\begin{cases}-2x+y+z=0\\ x-y+z=0\\ x+y-2z=0\end{cases}\Leftrightarrow\begin{cases}-2x+y+z=0\\ -3x+3z=0\\ x+y-2z=0\end{cases} \\
&\Leftrightarrow x=y=z,
\end{aligned}$$

So

$$\begin{aligned}
\ker(f) &= \left\{(x,y,z)\in\mathbb{R}^3: x=y=z\right\} \\
&= \left\{x\underbrace{(1,1,1)}_{u_1}: x\in\mathbb{R}\right\}
\end{aligned}$$

then $\{u_1\}$ is basis of $\ker(f)$, and then

$$\dim\left(\operatorname{Im}\left(f\right)\right) = \dim \mathbb{R}^3 - \dim\ker(f) = 3 - Card\left\{u_1\right\} = 2$$

(3) We have

$$\begin{aligned} \operatorname{Im}(f) &= \left\{f\left(x,y,z\right) : (x,y,z) \in \mathbb{R}^3\right\} \\ &= \left\{x\underbrace{(-2,1;1)}_{v_1} + y\underbrace{(1-2,1)}_{v_2} + z\underbrace{(1,1,-2)}_{v_3} : (x,y,z) \in \mathbb{R}^3\right\} \end{aligned}$$

Then $\{v_{1,}v_2,v_3\}$ is a generates family, as $v_1 + v_2 = -v_3$ and $\dim\left(\operatorname{Im}(f)\right) = 2$, then $\{v_{1,}v_2\}$ is generates family, show that $\{v_{2,}v_3\}$ is free. Let $\lambda_2, \lambda_3 \in \mathbb{R}$,

$$\begin{aligned} \lambda_2 v_2 + \lambda_3 v_3 &= 0_{\mathbb{R}^3} \Rightarrow (-2\lambda_2 + \lambda_3, \lambda_2 - \lambda_3, \lambda_2 + \lambda_3) = (0,0,0) \\ &\Rightarrow \lambda_2 = \lambda_3 = 0 \end{aligned}$$

so $\{v_{2,}v_3\}$ is a basis of $\operatorname{Im}(f)$.