Series 2 : Sequences of real numbers

Exercise 1 : 1. Study the convergence of the following sequences :

a. $U_n = \sqrt{n^2+n+1} - \sqrt{n}$ b. $U_n = \dfrac{3^n + (-3)^n}{3^n}$ c. $U_n = \left(1 + \dfrac{2}{n}\right)^n$

2. Show that the following sequence is bounded.

$$U_n = \frac{2 + cos(n)}{3 - \sin\sqrt{n}}, \quad \forall n \in \mathbb{N}$$

Exercise 2 : Consider the sequence (U_n) defined by :

$$\begin{cases} U_0 = \dfrac{3}{2} \\ U_{n+1} = (U_n - 1)^2 + 1 \end{cases}$$

1. Show that : $1 < U_n < 2$, $\forall\ n \in \mathbb{N}$.
2. Study the monotonicity of $(U_n)_{n\in\mathbb{N}}$.
3. Deduce that (U_n) is convergent and determine its limit.

Exercise 3 : Let the sequence $(U_n)_{n\in\mathbb{N}^*}$ be defined by :

$$U_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$$

Consider two sequences (V_n) and (W_n) defined by : $V_n = U_{2n}, \quad W_n = U_{2n+1}$.

1. Show that (V_n) and (W_n) are adjacent.
2. Deduce that the sequence (U_n) is convergent.

Exercise 4 : Consider the sequences (U_n) and (V_n), $n \in \mathbb{N}$, defined by :

$$\begin{cases} U_0 = 2, \quad U_1 = \dfrac{4}{9}, \\ U_{n+2} = \dfrac{1}{27}(12U_{n+1} - U_n), \end{cases}$$

and $V_n = U_n - \dfrac{1}{3^n}, \quad \forall n \in \mathbb{N}$.

1. Show that for all $n \in \mathbb{N}$: $U_{n+1} = \dfrac{1}{9}U_n + \dfrac{2}{3^{n+2}}$.
2. Show that the sequence (V_n) is a geometric sequence, then write U_n in terms of n
3. Write in terms of n the sum : $S_n = \sum_{k=0}^{n} U_k$