Series 2 : Real functions of a real variable.

Exercise 1 :

Calculate the following limits :

1) $\lim\limits_{x\to+\infty} \dfrac{1+\cos x}{\sqrt{x}}$ 2) $\lim\limits_{x\to 0} \dfrac{\ln(1+x^2)}{\sin^2 x}$

3) $\lim\limits_{x\to 0} x \exp\left(\dfrac{1}{x}-1\right)$ 4) $\lim\limits_{x\to+\infty} \left(\dfrac{x}{x-2}\right)^x$.

Exercise 2 :

Using the l'Hospital's rule to calculate :

$\lim\limits_{x\to 0} \dfrac{2\sin x - \sin 2x}{x - \sin x.}$ $\lim\limits_{x\to 0} \dfrac{e^{4x}-e^{x}}{\tan x}$

Exercise 3 :

1. Let $f(x) = \dfrac{\sqrt{3x^2+1}-2}{x-1}$ be the function.

   a - Calculate the limit of f at 1.
   b - Deduce a function g which is an extension by continuity of f at 1.

2. Are the following functions extendable by continuity at 0 ?
   $f(x) = x^2 \sin \dfrac{1}{x}$ $g(x) = \dfrac{\sin^2 x}{x}$.

Exercise 4 :

1. Determine a real α such that the following function is continuous on R :

$$f(x) = \begin{cases} e^{-x}+1 & x<0 \\ \alpha & x=0 \\ 2-x\ln x & x>0. \end{cases}$$

2. With the value α found , is the function differentible at 0 ?

Exercise 5 :

Let f be a function defined by :

$$f(x) = \begin{cases} \dfrac{1-\cos(2\pi x)}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that there exists $c \in]-1,1[$ such that $f'(c) = 0$.

Exercise 6 :

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by :

$$f(x) = \begin{cases} 3 - \dfrac{x^2}{2} & \text{if } x \geq 1, \\ \dfrac{1}{x} & \text{if } x < 1. \end{cases}$$

1. Are the hypotheses of the mean value theorem satisfied for the function f on the interval [0,2] ?
2. If so, find the corresponding intermediate values.

Additional exercises

Exercise 7 :

Let $f : \mathbb{R}/\{-1, 1\} \to \mathbb{R}$ be a function defined by :

$$f(x) = \frac{x^3 - 2x^2 - x + 2}{1 - \mid x \mid}$$

1. Is the function f continious at 0 ?
2. Calculate $\lim\limits_{x\to-1} f(x)$ and $\lim\limits_{x\to 1} f(x)$.

Exercise 8 :

For each of the following cases, determine α so that the function is continuous on $\mathbb{R}$.

1) $f(x) = \begin{cases} \dfrac{x^2 - x}{x} & x \neq 0 \\ \alpha & x = 0. \end{cases}$ 2) $f(x) = \begin{cases} \dfrac{\sqrt{x^2 - x} + 1 - x}{x} & x \neq 1 \\ \alpha & x = 1. \end{cases}$

Exercise 9 :

The functions $f, g : \longrightarrow \mathbb{R}$ defined by :

$$f(x) = |x| \sin x \qquad\qquad g(x) = \ln(1 + |x|)$$

are they differentiable at 0.