University of Badji-Mokhtar Annaba
Faculty of Technologie
Computer Science Department

Annaba 22 /01/2025

1h30

Exam Mathematical Analysis 1

Exercise 1 ($6pts$)

Answer with true or false, justifying your answer:

- If $\sup A = M \Leftrightarrow \begin{cases} \forall x \in A; x \leq M \\ \forall \varepsilon > 0; \exists\ x_0 \in A : M + \varepsilon > x_0 \end{cases}$
- $(U_n)_{n\in\mathbb{N}}$ diverge to $+\infty$ iff $\exists N > 0, \exists A \in N, \forall n \geq N \Rightarrow U_n \leq A$.
- From any real sequence we can extract a convergent subsequence.
- The algebric equation $z^2 - (1+2i)\,z + i - 3 = 0$ have 2 solutions $z_1 = -i$ and $z_2 = 1 + i$
- $\lim\limits_{x\to 0} \dfrac{\ln(1+(\sin x)^2)}{\tan \frac{x}{2}} = -1$

Exercise 2 (6 pts)

Let the recurrent sequence $(U_n)_{n\in\mathbb{N}^*}$ defined by

$$\begin{cases} U_1 = \dfrac{1}{2} \\ U_{n+1} = U_n^2 + \dfrac{3}{16}, n \geq 1 \end{cases}$$

1. Show that $\forall n \geq 1, \dfrac{1}{4} < U_n < \dfrac{3}{4}$. .
2. Show that $(U_n)_{n\in\mathbb{N}^*}$ is monotone
3. Deduct that the sequence $(U_n)_{n\in\mathbb{N}^*}$ converges and calculate its limit .
4. Let $E = \{U_n, n \geq 1\}$. Determine $\sup E$ and $\inf E$.

Exercise 3 ($8pts$)

We consider the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \ln x \text{ , if } x > 0 \\ x^2 \cos\left(\frac{1}{x}\right) \text{ , if } x < 0 \end{cases} \qquad f(0) = 0$$

a) Study the continuity and differentiability of the function f on $\mathbb{R}$

b) Is the function f of class C^1 on $\mathbb{R}$

"Calculators and cell phones are strictly prohibited"

Good Luck

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Département d'informatique

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Examen d'**Analyse 1**

Exercice 1 (6pts)

Répondre par vrai ou faux, en justifiant votre réponse:

- Si $\sup A = M \Leftrightarrow \begin{cases} \forall x \in A; x \leq M \\ \forall \varepsilon > 0; \exists\ x_0 \in A : M + \varepsilon > x_0 \end{cases}$
- $(U_n)_{n\in\mathbb{N}}$ diverge vers $+\infty$ ssi $\exists N > 0, \exists A \in N, \forall n \geq N \Rightarrow U_n \leq A$.
- De toute suite réelle, nous pouvons extraire une sous suite convergente.
- L'équation $z^2 - (1+2i)\, z + i - 3 = 0$ admet 2 solutions $z_1 = -i$ et $z_2 = 1 + i$
- $\lim\limits_{x\to 0} \dfrac{\ln(1 + (\sin x)^2)}{\tan \frac{x}{2}} = -1$

Exercice 2 (6pts)

Soit la suite récurrente $(U_n)_{n\in\mathbb{N}^*}$ définie par

$$\begin{cases} U_1 = \dfrac{1}{2} \\ U_{n+1} = U_n^2 + \dfrac{3}{16}, n \geq 1 \end{cases}$$

1. Montrer que $\forall n \geq 1, \dfrac{1}{4} < U_n < \dfrac{3}{4}$.
2. Montrer que la suite $(U_n)_{n\in\mathbb{N}^*}$ est monotone
3. Déduire que la suite $(U_n)_{n\in\mathbb{N}^*}$ converge puis calculer sa limite
4. Soit $E = \{U_n, n \geq 1\}$. Déterminer $\sup E$ et $\inf E$.

Exercice 3 (8pts)

1-On considère la fonction f définie sur $\mathbb{R}$ par

$$f(x) = \begin{cases} x^2 \ln x \text{ , si } x > 0 \\ x^2 \cos\left(\frac{1}{x}\right) \quad \text{, si } x < 0 \end{cases} \text{ avec } f(0) = 0$$

a) Etudier la continuité et dérivabilité de la fonction f sur $\mathbb{R}$

b) La fonction f est-t elle de classe C^1 sur $\mathbb{R}$

"La calculatrice et le téléphone portable sont strictement interdits"

Bon courage

Corrected

Exercise 1 $(6pts)$

1) Answer with true or false

• If $\sup A = M \Leftrightarrow \begin{cases} \forall x \in A; x \leq M \\ \forall \varepsilon > 0; \exists\, x_0 \in A : M + \varepsilon > x_0 \end{cases}$ →. FALSE → 0.25pt

$\sup A = M \Leftrightarrow \begin{cases} \forall x \in A; x \leq M \\ \forall \varepsilon > 0; \exists\, x_0 \in A : M + \varepsilon < x_0 \leq M \end{cases}$ → 0.75pt

•From any real sequence we can extract a convergent subsequence. FALSE → 0.25pt

conter - example $U_n = 2^n$ /

we can extract a convergent subsequence if the sequence is bounded → 0.75pt

• $(U_n)_{n\in\mathbb{N}}$ diverge to $+\infty$ iff $\exists N > 0, \exists A \in N, \forall n \geq N \Rightarrow U_n \leq A$. FALSE → 0.25pt

$(U_n)_{n\in\mathbb{N}}$ diverge to $+\infty$ iff $\forall A > 0, \exists N \in N, \forall n \geq N \Rightarrow U_n > A$ → 0.75pt

•$\lim\limits_{x\to 0} \dfrac{\ln(1 + (\sin x)^2)}{\tan \frac{x}{2}} = -1$ FALSE → 0.25pt

$$\lim_{x\to 0} \frac{\ln(1 + (\sin x)^2)}{\tan \frac{x}{2}} \sim \lim_{x\to 0} \frac{(\sin x)^2}{\tan \frac{x}{2}} \sim \lim_{x\to 0} \frac{x^2}{\frac{x}{2}} \sim \lim_{x\to 0} 2x = 0 \rightarrow \boxed{1\text{pt}}$$

•$z^2 - (1 + 2i)\, z + i - 3 = 0$ have 2 solutions $z_1 = -i$ and $z_2 = 1 + i$ FALSE →→ 0.25pt

$\Delta = (1 + 2i)^2 - 4\,(i - 3) = 1 + 4i - 4 - 4i + 12 = 9 > 0$ 0.5 pt

then $\begin{cases} z_1 = \dfrac{1 + 2i - 3}{2} = -1 + i & \boxed{0.5\text{ pt}} \\ z_1 = \dfrac{1 + 2i + 3}{2} = 2 + i & \boxed{0.5\text{ pt}} \end{cases}$

Exercise 2 $(6pts)$

1) Show that for $n \geq 1, \dfrac{1}{4} \leq U_n \leq \dfrac{3}{4}$

by induction: for $n = 1 \Rightarrow U_1 = \dfrac{1}{2} \left(\dfrac{1}{4} < U_1 < \dfrac{3}{4}\right)$ *true* → 0.5 pt

we suppose that $\dfrac{1}{4} < U_n < \dfrac{3}{4}$, and we prove that $\dfrac{1}{4} < U_{n+1} < \dfrac{3}{4}$

we have: $\dfrac{1}{4} < U_n < \frac{3}{4} \Rightarrow \quad \dfrac{1}{16} < U_n^2 < \dfrac{9}{16}$

$$\Rightarrow \quad \frac{1}{16} + \frac{3}{16} < U_n^2 + \frac{3}{16} < \frac{9}{16} + \frac{3}{16}$$

$$\Rightarrow \frac{1}{4} < U_n^2 + \frac{3}{16} < \frac{3}{4} \Rightarrow \frac{1}{4} < U_{n+1} < \frac{3}{4}$$

Then for $n \geq 1, \dfrac{1}{4} < U_n < \dfrac{3}{4}$ → 1 pt

2) Study the monotonic

$$U_{n+1}-U_n=U_n^2+\frac{3}{16}-U_n$$

study the sign of $\quad U_n^2-U_n+\dfrac{3}{16}$

$$\triangle=1-4.\frac{3}{16}=\frac{1}{4}>0\Rightarrow U_{n1}=\frac{3}{4},U_{n2}=\frac{1}{4}\longrightarrow\boxed{0.5\text{ pt}}$$

we have for $n\geq 1,\dfrac{1}{4}<U_n<\dfrac{3}{4}\Rightarrow U_n^2-U_n+\dfrac{3}{16}<0\Rightarrow\quad U_{n+1}-U_n<0\ \longrightarrow\boxed{0.5\text{ pt}}$

Then $\quad U_n$ is decreasing $\rightarrow\boxed{0.5\text{ pt}}$

3. Since (U_n) is decreasing and lower bounded by $\dfrac{1}{4}$, then (U_n) converges to the limit l $\rightarrow\boxed{1\text{pt}}$

this limit verify $\quad \lim\limits_{n\to+\infty}U_{n+1}=\lim\limits_{n\to+\infty}U_n=l\Rightarrow l=l^2+\dfrac{3}{16}\Rightarrow l^2-l+\dfrac{3}{16}=0\quad\Rightarrow l_1=\dfrac{3}{4}$

or $\quad l_2=\dfrac{1}{4}\longrightarrow\boxed{0.5\text{ pt}}$

since the sequence is decreasing and $U_1=\dfrac{1}{2}$ then $l=l_2=\dfrac{1}{4}\longrightarrow\boxed{0.5\text{ pt}}$

4.Since the sequence is decreasing then $\sup E=\max E=U_1=\dfrac{1}{2}\longrightarrow\boxed{0.5\text{ pt}}$

and $\inf E=\lim U_n=\dfrac{1}{4}\longrightarrow\boxed{0.5\text{ pt}}$

Exercise 3 (8*pts*)

The continuity and differentiability of f on $\mathbb{R}$

a)•We have $f(x)$ is continuous on $]-\infty,0[\cup]0,+\infty[$

$$\text{because}\begin{cases}x^2\ln x\text{ is continous on }]0,+\infty[\\ x^2\cos\left(\frac{1}{x}\right)\text{ is continuous on }]-\infty,0[\end{cases}\longrightarrow\boxed{0.5\text{pt}}$$

$f(x)$ is continuous at $x_0=0$ iff

$\lim\limits_{x\overset{<}{\to}0}f(x)=\lim\limits_{x\overset{>}{\to}0}f(x)=f(0)\rightarrow\boxed{0.25\text{pt}}\quad\Rightarrow\lim\limits_{x\overset{<}{\to}0}x^2\cos\left(\dfrac{1}{x}\right)=0=f(0)\Rightarrow f(x)$ is continuous at left $x_0=0\rightarrow\boxed{0.75\text{pt}}$

$\lim\limits_{x\overset{>}{\to}0}f(x)=\lim\limits_{x\overset{>}{\to}0}x^2\ln x=0=f(0)$ is continuous at left $x_0=0\rightarrow\boxed{0.75\text{pt}}$

we have $\lim\limits_{x\overset{<}{\to}0}f(x)=\lim\limits_{x\overset{>}{\to}0}f(x)=f(0)$, then $f(x)$ is continuous at $x_0=0\rightarrow\boxed{0.5\text{pt}}$

So $f(x)$ is continuous on $\mathbb{R}\rightarrow\boxed{0.25\text{pt}}$

•We have $f(x)$ is differentiable on $]-\infty,0[\cup]0,+\infty[$

$$\text{because}\begin{cases}x^2\ln x\text{ is differentiable on }]-\infty,0[\\ x^2\cos\left(\frac{1}{x}\right)\text{ is differentiable on }]0,+\infty[\end{cases}\rightarrow\boxed{0.5\text{pt}}$$

$\Rightarrow$then f is right-hand differentiable at $x_0 = 0 \rightarrow$ $\boxed{0.75\text{pt}}$

we have $f_l'(0) = f_r'(0)$ $\rightarrow$ $\boxed{0.25\text{pt}}$ hence is f is differentiable on $\mathbb{R} \rightarrow$ $\boxed{0.25\text{pt}}$

2- $f'(x) = \begin{cases} 2x\ln x + x \text{ , if } x < 0 \\ f'(0) = 0 \\ 2x\cos\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x}\right) \quad \text{, if } x > 0 \end{cases}$ $\rightarrow$ $\boxed{0.5\text{pt}}$

$f'(x)$ is from class 1 (C^1) if $f'(x)$ is continuous ie $\lim\limits_{x \overset{<}{\to} 0} f'(x) = \lim\limits_{x \overset{>}{\to} 0} f'(x) = f'(0) \rightarrow$ $\boxed{0.25\text{pt}}$

$\lim\limits_{x \overset{<}{\to} 0} f'(x) = \lim\limits_{x \overset{<}{\to} 0} 2x\ln x + x = 0 \rightarrow$ $\boxed{0.5\text{pt}}$

$\lim\limits_{x \overset{>}{\to} 0} 2x\cos\left(\dfrac{1}{x}\right) - \sin\left(\dfrac{1}{x}\right)$ = the limit does not exist$\rightarrow$ $\boxed{0.5\text{pt}}$

then $f'(x)$ is not continuous at $x_0 = 0 \Rightarrow f$ does not from $C^1 \rightarrow$ $\boxed{0.25\text{pt}}$