Algebra 1 Course

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Algebra 1

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Coefficient : 03

Credits : 05

Time per week : 01 :30 hours lessons and 01 :30 hours TD

Evaluation method : Exam$\times 0,6$ $+$ **TD**$\times 0,4$.

Lesson Plan :

1. Mathematical Logic.
2. Sets, Applications and Binary relations.
3. Algebraic structures.
4. Polynomials.

References :

1. M. Mignotte et J. Nervi, Algèbre : licences sciences 1ère année, Ellipses, Paris, 2004.
2. J. Franchini et J. C. Jacquens, Algèbre : cours, exercices corrigés, travaux dirigés, Ellipses, Paris, 1996.
3. C. Degrave et D. Degrave, Algèbre 1ère année : cours, méthodes, exercices résolus, Bréal, 2003.
4. S. Balac et F. Sturm, Algèbre et analyse : cours de mathématiques de première année avec exercices corrigés, Presses Polytechniques et Universitaires romandes, 2003.

CHAPITRE 1

MATHEMATICAL LOGIC

1.1 Logic

1.1.1 Concept of proposition

Définition 1.1. *A proposition is a mathematical expression which is either true or false, and it cannot be true and false at the same time.*

Exemple 1.1.
1. "$1+2=3$ " *is a true proposition.*
2. "5 ***is an even number"*** *is a false proposition.*
3. "$\sqrt{2}$ ***is not a rational number."*** *is a true proposition.*
4. " n ***divides*** 8 " *is not a proposition, because it depends on the value of* n.

Remarque 1.1.
- *A proposition is usually denoted by a capital letter such as* ***P****,* ***Q****, or* ***R****."*
- *We write* 1 *for a true proposition and* 0 *for a false proposition.*
- *For several propositions, we use a truth table to show all possibilities.*

Exemple 1.2.
1. *For one proposition, there are two possibilities :*

P
1
0

2. *For two propositions, there are four possibilities :*

P	Q
1	1
1	0
0	1
0	0

1.1.2 The negation of a proposition

Définition 1.2. *The negation of* $\boldsymbol{P}$ *is written as* $\neg \boldsymbol{P}$ *or* $\bar{\boldsymbol{P}}$. *It is true when* $\boldsymbol{P}$ *is false, and false when* $\boldsymbol{P}$ *is true. The truth table is :*

P	$\bar{P}$
1	0
0	1

TABLE 1.1 – Table de vérité de $\bar{P}$

Exemple 1.3. 1. P :**"2 is a prime number"**. *(true proposition)*
$\bar{P}$: **" 2 is not a prime number "**. *(false proposition)*

2. Q : "$3 > 5$".*(false proposition)*
$\bar{Q}$: "$3 \leq 5$". *(true proposition)*

Remarque 1.2. *The negation of the negation of* $\boldsymbol{P}$ *is the same as* $\boldsymbol{P}$. *So :* $\bar{\bar{P}} = P$

1.1.3 Logical connectives

Let **P** and **Q** be two logical propositions.

The conjunction

Définition 1.3. *"The conjunction of* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *is written* $(P \wedge Q)$. *It is true only when both* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *are true. The truth table is :*

P	Q	$P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

TABLE 1.2 – Table de vérité de $P \wedge Q$

Exemple 1.4. *P : " 2 is an even number " (true)*
Q : " 2 is a prime number "(true)
Then : $P \wedge Q$ *: "2 is an even number and 2 is a prime number" (true).*

The disjunction

Définition 1.4. *The disjunction of* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *is written* $(P \vee Q)$. *It is true if at least one of* $\boldsymbol{P}$ *or* $\boldsymbol{Q}$ *is true. It is false only if both are false. The truth table is :*

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

TABLE 1.3 – Table de vérité de $P \vee Q$

Exemple 1.5. *P : " 4 is a multiple of 2 " (true)*
Q : " 4 is a multiple of 3 "(false)
Then : $P \vee Q$: " 4 is a multiple of 2 or 4 is a multiple of 3 " (true).

Implication

Définition 1.5. *The implication of* ***P*** *and* ***Q*** *is written $(P \Rightarrow Q)$. It means "If* ***P*** *then* ***Q***". *It is false only when* ***P*** *is true and* ***Q*** *is false ; in all other cases it is true. The truth table is :*

P	Q	$P \Rightarrow Q$
1	1	1
1	0	0
0	1	1
0	0	1

TABLE 1.4 – Table de vérité de $P \Rightarrow Q$

Exemple 1.6. *P : " $x = 2$ "*
Q : " $x^2 = 4$ "
$P \Rightarrow Q$: "if $x = 2$, then $x^2 = 4$ " (true).

Remarque 1.3. *The implication $(P \Rightarrow Q)$ is equivalent to the proposition $\left(\bar{P} \vee Q\right)$.*

Equivalence

Définition 1.6. *The equivalence of two propositions* ***P*** *and* ***Q*** *is the proposition writen $(P \Leftrightarrow Q)$. It is true when* ***P*** *and* ***Q*** *are both true or both false, and false in all other cases. It can also be defined as :*

$$(P \Leftrightarrow Q) \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

The truth table is :

P	Q	$P \Leftrightarrow Q$
1	1	1
1	0	0
0	1	0
0	0	1

TABLE 1.5 – Table de vérité de $P \Leftrightarrow Q$

Exemple 1.7. *P : " x is even. "*
Q : " x is divisible by 2. "
$P \Leftrightarrow Q$: " x is even if and only if x is divisible by 2 " (true).

Propriété 1.1. *Let P, Q and R be three logical propositions, then :*

1. *Commutativity of $\wedge$ and $\vee$:*
$$P \wedge Q \Leftrightarrow Q \wedge P$$
$$P \vee Q \Leftrightarrow Q \vee P$$
2. *Associativity of of $\wedge$ and $\vee$:*
$$(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$$
$$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$$

3. *Distributivity*
$$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge Q)$$
$$P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee Q)$$

4. *De Morgan's Laws*
$$\overline{(P \wedge Q)} \Leftrightarrow (\bar{P} \vee \bar{Q})$$
$$\overline{(P \vee Q)} \Leftrightarrow (\bar{P} \wedge \bar{Q})$$

1.1.4 The Quantifiers

Universel quantifier "$\forall$"

— **Symbol :** $\forall$ (read as "for all" or "for every").
$$\forall x \in E, \quad P(x)$$

— **Meaning :** The proposition $P(x)$ holds for all elements of the set E.

Exemple 1.8. *The proposition :* $\forall z \in \mathbb{C}, |z| = 1$ *(false), because for* $z = 2i$ *we have* $|z| = 2 \neq 1$.

Existential quantifier "$\exists$"

— **Symbol :** $\exists$ (read as "there exists").
$$\exists x \in E, P(x)$$

— **Meaning :** There is at least one element in the set E that satisfies the proposition $P(x)$.

Exemple 1.9. *The proposition :* $\exists x \in \mathbb{R}, x + 2 > 5$ *(true), because for* $x = 4$ *we have* $x + 2 = 6 > 5$.

Remarque 1.4. — ***Symbol :*** $\exists!$ *(read as "There exists a unique").*
$$\exists! x \in E, \quad P(x)$$

— ***Meaning :*** *There exists a unique element in the set* E *that satisfie the proposition* $P(x)$.

Exemple 1.10. *The proposition :* $\exists! x \in \mathbb{R}, x^2 = 0$ *(true), because for* $x = 0$ *we have* $0^2 = 0$.

the negation of quantifiers

1. **Negation of the universal quantifier** " $\forall x \in E, P(x)$ " is : " $\exists x \in E, \bar{P}(x)$ ".
2. **Negation of the existential quantifie** " $\exists x \in E, P(x)$ " is : " $\forall x \in E, \bar{P}(x)$ ".

Exemple 1.11.
1. *The negation of the proposition* " $\forall x \in \mathbb{R}, e^x > 0$ "*(true)*
is : " $\exists x \in \mathbb{R}, e^x \leq 0$ " *(false).*
2. *The negation of the proposition* " $\exists x \in \mathbb{R}, x + 2 = 3$ "*(true)*
is : " $\forall x \in \mathbb{R}, x + 2 \neq 3$ "*(false).*

1.2 Types of Reasoning

In this section, we present the different types of mathematical reasoning.

1.2.1 Direct reasoning

To prove $(P \Rightarrow Q)$ is true.

1. Assume that p is true.
2. Use p to show that q must be true.

Exemple 1.12. *Show that if $n \in \mathbb{N}$ is even, then $\Rightarrow n^2$ is even.*

— *Assume that n is even, i.e., $\exists k \in \mathbb{Z}, n = 2k$.*
— *Then*

$$n^2 = 2(2k^2) \Rightarrow n^2 = 2k'$$

with $k' = 2k^2 \in \mathbb{Z}$ so $\exists k' \in \mathbb{Z}, n^2 = 2k', n^2$ is even.

1.2.2 Reasoning by contrapositive

Reasoning by contrapositive is based on the following equivalence : $(P \Rightarrow Q) \Leftrightarrow \left(\bar{Q} \Rightarrow \bar{P}\right)$. So, to prove that $(P \Rightarrow Q)$ is true, it is enough to prove the contrapositive : $\left(\bar{Q} \Rightarrow \bar{P}\right)$ directly : we assume $\bar{Q}$ is true and then show that $\bar{P}$ is true.

Exemple 1.13. *Show that : if n^2 is odd, then $\Rightarrow n$ is odd.*
— *Contrapositive : if n is even, then $\Rightarrow n^2$ is even.*
— *Easy to prove (see previous example) $\rightarrow$ so the original statement is true.*

1.2.3 Proof by contradiction

To prove $(P \Rightarrow Q)$ by contradiction : assume P is true and Q is false, and show that this leads to a contradiction.

Exemple 1.14. *Let $a, b > 0$. Show that : $\frac{a}{1+b} = \frac{b}{1+a} \Rightarrow a = b$.*

Démonstration. Assume that : $\frac{a}{1+b} = \frac{b}{1+a}$ and $a \neq b$.
then :

$$\begin{aligned} a(1+a) = b(1+b) \text{ and } a \neq b &\Rightarrow a - b + a^2 - b^2 = 0 \text{ and } a \neq b \\ &\Rightarrow (a-b)(a+b) = 0 \text{ and } a \neq b \\ &\Rightarrow (a = b) \text{ because } (a+b) > 0 \text{ and } a \neq b \end{aligned}$$

We get a contradiction. Then : $\frac{a}{1+b} = \frac{b}{1+a} \Rightarrow a = b$. □

1.2.4 Proof by Induction

To prove $P(n) : \forall n \in \mathbb{N}, n \geq n_0$, do the following : :

1. Prove $P(n_0)$ is true.
2. Assume $P(n)$ is true for an arbitrary $n \geq n_0$.
3. Using the hypothesis, prove $P(n+1)$ is true.
   Then : by induction, $P(n)$ holds for every $n \geq n_0$.

Exemple 1.15. *Show that : $\forall n \in \mathbb{N}^*, 1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$*

1. *For $n = 1, P(1)$ is true $1 = \frac{1(2)}{2}$.*

2. *Assume that :* $1+2+\ldots+n=\dfrac{n(n+1)}{2}$ *is true.*

3. *Show that :* $1+2+\ldots+(n+1)=\dfrac{(n+1)(n+2)}{2}$ *is true.*
   We have :

$$\begin{aligned} 1+2+\ldots+n+(n+1) &= \frac{n(n+1)}{2}+(n+1) \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

   Thus $P(n+1)$ *is true.*
   Then : by induction, $\forall n \in \mathbb{N}^*, 1+2+\ldots+n=\dfrac{n(n+1)}{2}$ *is true.*