Series of Exercises 01
Mathematical Logic and Proofs

Exercice 1 *Among the following expressions, which ones are propositions ? For propositions, indicate whether they are true or false.*

1. 11 *is a prime number.*
2. $((-5)^2 = 25) \quad \wedge \quad \left(\sqrt{25} = -5\right)$.
3. $\left(\sqrt{3}\ \text{ is an integer}\right) \quad \vee \quad (\text{all the sides of a square are equal})$.
4. $\forall x \in \mathbb{R}, \quad x + 2 > 5$.
5. $\exists z \in \mathbb{C}, \quad |z| = 1$.
6. $x \in \mathbb{N}$.

Exercice 2 *Let P, Q and R be propositions. Using truth table, show the follwing relations :*

1. $\overline{(P \wedge Q)} \Leftrightarrow (\bar{P} \vee \bar{Q})$
2. $\overline{(P \Rightarrow Q)} \Leftrightarrow (P \wedge \bar{Q})$ ***(homework)***
3. $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge Q)$
4. $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee Q)$ ***(homework)***.

Exercice 3 *Consider the following four propositions :*

1. $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : x + y > 0$.
2. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x + y > 0$.
3. $\forall x \in \mathbb{R}, \forall y \in \mathbb{R} : x + y > 0$.
4. $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : y^2 > x$.
   Are these propositions true or false ? Provide their negations.

Exercice 4 *Let f be a function from $\mathbb{R}$ to $\mathbb{R}$, write in terms of quantifiers the following expressions :*

1. *f is positive.*
2. *f is even.*
3. *f is strictly increasing function.*
4. *f is bounded above by a constant M.*

Exercice 5

1. *Let a and b be two real numbers. Using direct reasoning, show that :*
$$a^2 + b^2 = 1 \Rightarrow |a + b| \leq 2$$
2. *Let $n \in \mathbb{N}$, by contrapositive, Prove that :*
$$n^2 \text{ is divisible by } 3 \Rightarrow n \text{ is divisible by } 3.$$
3. *Using the proof by contradiction prove that :*
   a) *$\sqrt{3}$ is an irrational number.*
   b) $\forall x \in \mathbb{R}^* : \quad \sqrt{1 + x^2} \neq 1 + \frac{x^2}{2}$.***(homework)***
4. *Prove by induction :*
   a) $\forall n \in \mathbb{N}^*, 1 + 2^3 + 3^3 + \ldots + n^3 = \dfrac{n^2(n+1)^2}{4}$
   b) $\forall n \in \mathbb{N} : (21^n - 2^{2n})$ *is divisible by* 17. ***(homework)***