Series of exercises 1: the Field of real numbers

Exercise 1:

a) Prove that $\forall n > 1 : n^n \geq n!$

b) Let $(a,b) \in \mathbb{Q}^+ \times \mathbb{Q}^+$ such that $\sqrt{ab} \notin \mathbb{Q}$

Prove that $\sqrt{a} + 2\sqrt{b} \notin \mathbb{Q}$ and $\dfrac{\ln 2}{\ln 3} \notin Q$ (are irrational numbers).

Exercise 2:

1).Solve the following equation

$$E\,(2x-1)+1=0$$

2) Show the following properties :

•$\forall x \in \mathbb{R}, \forall n \in \mathbb{N}^* :\ E(\dfrac{E(nx)}{n}) = E(x).$

• $\forall x \in \mathbb{R} :\ E\left(\dfrac{x}{2}\right) + E\left(\dfrac{x+1}{2}\right) = E(x)$

Exercise 3:

1) Prove the following inequalitie:

• $\forall x, y \in \mathbb{R},\ \ ||x| - |y|| \leq |x+y|$

• $\forall\ x, y \in \mathbb{R} : \max(x,y) = \dfrac{x+y+|x-y|}{2}$

$$\min(x,y) = \frac{x+y-|x-y|}{2}$$

Exercise 4:

Let $A = \{x \in \mathbb{R} : |2x+1| \leq 5\}$; $B = \{x \in \mathbb{R}, e^x < \frac{1}{2}\}$,

$$C = \{x \in \mathbb{Q} : x^2 < 2\}.$$

Are A, B and C bounded below? bounded above?, do they admit Inf, Min? Sup, Max?

Exercise 5:

Let A be a non -empty and bounded subset of $\mathbb{R}$.

We denote $B = \{|x-y|\,; (x, y \in A^2)\}$.

1.Justify that B is bounded above.

2. We denote $\sup(B)$ as the supremum of the set B and $\inf(A)$ as infimum of the set A

Show that $\sup(B) = \sup(A) - \inf(A)$.

Exercise 6:

Determine the minimum, maximum, Supremum and infimum if they exist while justifying the answers of the following sets :

$$A = \{ \frac{2n-3}{n+1}; n \in \mathbb{N} \}, \quad B = \left\{ \frac{1}{n} + \frac{1}{n^2} / \quad n \in \mathbb{N}^* \right\},$$

$$C = \{x^2 + 1; x \in]1, 2]\}$$