Algebra 1 Course

Neggal Bilel

13 octobre 2025

Algebra 1

Teacher of lessons : Dr. Neggal Bilel

Contact : bilel.neggal@univ-annaba.dz

Coefficient : 03

Credits : 05

Time per week : 01 :30 hours lessons and 01 :30 hours TD

Evaluation method : Exam$\times 0,6$ $+$ **TD**$\times 0,4$**.**

Lesson Plan :

1. Mathematical Logic.
2. Sets, Applications and Binary relations.
3. Algebraic structures.
4. Polynomials.

References :

1. M. Mignotte et J. Nervi, Algèbre : licences sciences 1ère année, Ellipses, Paris, 2004.
2. J. Franchini et J. C. Jacquens, Algèbre : cours, exercices corrigés, travaux dirigés, Ellipses, Paris, 1996.
3. C. Degrave et D. Degrave, Algèbre 1ère année : cours, méthodes, exercices résolus, Bréal, 2003.
4. S. Balac et F. Sturm, Algèbre et analyse : cours de mathématiques de première année avec exercices corrigés, Presses Polytechniques et Universitaires romandes, 2003.

CHAPITRE 1
SETS, FUNCTIONS, AND BINARY RELATIONS

1.1 Sets

Definition 1.1. *A set is a collection of objects called the elements of that set.*

Notations :

- n general, sets are represented by capital letters : $E, F, ...$
- Elements of a set are usually represented by lowercase letters : $x, y, ...$
- If x is in E, we write $x \in E$.
- If x is not in E, we write $x \notin E$.

Example 1.1. 1. *The set of digits in the decimal system :*

$$E = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\},$$

we have : $5 \in E, \quad 15 \notin E$.

2. *The set of even natural numbers :*

$$\mathbb{P} = \{x \in \mathbb{N} \text{ such that } 2 \text{ divides } x\},$$

we have : $2 \in \mathbb{P}, \quad 3 \notin \mathbb{P}$.

3. *The empty set is denoted by* $\emptyset$*, which contains no elements.*
4. *The set*

$$F = \{x \in \mathbb{R} \text{ such that } |x - 1| \leq 2\} = [-1, 3].$$

1.1.1 Sets Inclusion

Definition 1.2. *Let A and B be two sets. We say that A is included in B if and only if every element of A is in B, and we write $A \subset B$.*

Remark 1.1. 1. *If A is included in B, we can say that A is a part of B (or A is a subset of B), and we write :*

$$A \subset B \Leftrightarrow \forall x : x \in A \Rightarrow x \in B.$$

2. *We say that A is not included in B if there exists at least one element of A that does not belong to B, and we write :*

$$A \not\subset B \Leftrightarrow \exists x : x \in A \wedge x \notin B.$$

Example 1.2. 1. *We have :* $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

2. *If* $A =]1, 2[$ *and* $B = [1, 2]$*, then* $A \subset B$

1.1.2 Equality of Two Sets

Definition 1.3. *Let A and B be two sets. We say that A and B are equal if they have the same elements, and we write $A = B$.*

$$A = B \Leftrightarrow ((A \subset B) \wedge (B \subset A)).$$

Example 1.3. *Let*

$$A = \{x \in \mathbb{R} : |x-1| \leq 1\}, \quad B = [0,2], \quad C = [0,2[.$$

We have :

$$A = B,\ A \neq C,\ B \neq C,\ C \subset B.$$

1.1.3 Subsets of a Set

Definition 1.4. *Let E be a set. The set of all subsets of E is called the parts of a Set E, denoted $P(E)$.*

Example 1.4. *Let $E = \{-1, 0, 1\}$, then :*

$$P(E) = \{\emptyset, E, \{-1\}, \{0\}, \{1\}, \{-1,0\}, \{-1,1\}, \{0,1\}\}$$

1.1.4 Set Operations

Set Complement

Definition 1.5. *Let A be a subset of E. The complement of A in E is :*

$$C_E A = \{x \in E : x \notin A\}.$$

Example 1.5.
1. *Let $E = \{0,1,2,3,4,5\}$ and $A = \{1,3,5\}$, then : $C_E A = \{0,2,4\}$.*
2. *Let $I = [-1, 8[$, then : $C_{\mathbb{R}} I =]-\infty, -1[\cup [8, +\infty[$.*

Remark 1.2. $C_E(C_E A) = A$.

Intersection

Definition 1.6. *Let A and B be subsets of E. The intersection of A and B is :*

$$A \cap B = \{x \in E : x \in A \text{ and } x \in B\}.$$

Example 1.6.
1. *Let : $A = \{2,4,6\}$ and $B = \{3,4,5,6\}$, then : $A \cap B = \{4,6\}$.*
2. *Let : $A = [-5,5]$ and $A =]2,+\infty[$, then : $A \cap B =]2,5]$.*

Union

Definition 1.7. *Let A and B be subsets of E. The union of A and B is :*

$$A \cup B = \{x \in E : x \in A \text{ or } x \in B\}.$$

Example 1.7.
1. *Let : $A = \{0,2,4\}$ and $B = \{3,4,5,6\}$, then : $A \cup B = \{0,2,3,4,5,6\}$.*
2. *Let : $A = [-5,5]$ and $A =]2,+\infty[$, then : $A \cup B = [-5,+\infty[$.*

Remark 1.3.
1. *If A and B have no elements in common, they are said to be disjoint ; in this case, $A \cap B = \emptyset$.*
2. *$A \cap C_E A = \emptyset$ and $A \cup C_E A = E$.*

Difference of Two Sets

Definition 1.8. *Let A and B be subsets of E. The difference of A and B is :*

$$A \setminus B = \{x \in E : x \in A \text{ and } x \notin B\}.$$

Example 1.8. *Let : $E = \mathbb{R}, A = [-1, 2]$ and $B = [0, 3]$, then we have :*

$$A \setminus B = [-1, 0[\quad et \quad B \setminus A =]2, 3]$$

Symmetric Difference

Definition 1.9. *Let A and B be subsets of E. The symmetric difference of A and B is :*

$$A \vartriangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

Example 1.9. *Let : $E = \mathbb{Z}, A = \{-3, 0, 2, 6\}$ et $B = \{-1, 1, 2, 6\}$, then we have :*

$$A \vartriangle B = \{-3, -1, 0, 1\}$$

Properties of Set Operations

Let A, B, and C be subsets of a set E. We have :

1. **Commutativity :**
$$A \cap B = B \cap A, \quad A \cup B = B \cup A.$$
2. **Associativity :**
$$A \cap (B \cap C) = (A \cap B) \cap C, \quad A \cup (B \cup C) = (A \cup B) \cup C.$$
3. **Distributivity :**
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$
4. **De Morgan's Laws :**
$$(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c.$$

Démonstration. **6.** Show that : $(A \cup B)^c = A^c \cap B^c$ It must be shown that : $(A \cup B)^c \subset A^c \cap B^c$ and $A^c \cap B^c \subset (A \cup B)^c$.

— $(A \cup B)^c \subset A^c \cap B^c$:
Let $x \in (A \cup B)^c \Rightarrow x \notin (A \cup B) \Rightarrow x \notin A \wedge x \notin B \Rightarrow x \in A^c \wedge x \in B^c$ as well as $x \in (A \cup B)^c \Rightarrow x \in (A^c \cap B^c)$, from where $(A \cup B)^c \subset (A^c \cap B^c)$.

— $A^c \cap B^c \subset (A \cup B)^c$
Let $x \in (A^c \cap B^c) \Rightarrow x \in A^c \wedge x \in B^c \Rightarrow x \notin A \wedge x \notin B \Rightarrow x \notin (A \cup B), d'$ where $A^c \cap B^c \subset (A \cup B)^c$, then $(A \cup B)^c = A^c \cap B^c$.

□

1.1.5 Cartesian Product

Definition 1.10. *For two sets A and B, the Cartesian product is the set of all pairs (a, b) with $a \in A$ and $b \in B$:*

$$A \times B = \{(a, b) \mid a \in A,\, b \in B\}.$$

Example 1.10. *Let : $A = \{1, 2\}, B = \{1, 2, 3\}$*

$$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\},$$
$$B \times A = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\},$$
$$A \times B \neq B \times A, \ car \ (3,2) \in B \times A, \ and \ (3,2) \notin A \times B.$$

1.2 Functions

Definition 1.11. *A **function** from a set E to a set F is a correspondence f that associates to each element $x \in E$ one and only one element $y \in F$.*
The set E is called the domain, and F is called the codomain. The element y of F associated with an element x of E is denoted by $y = f(x)$.
$y = f(x)$ is called the image of x, and x is called a preimage (or antecedent) of y. We write :

$$\begin{aligned} f : E &\longrightarrow F \\ x &\longmapsto y = f(x) \end{aligned}$$

Example 1.11. *1. Consider the following functions :*

—
$$\begin{aligned} f : \mathbb{N} &\longrightarrow \mathbb{N} \\ n &\longmapsto 2n+1 \end{aligned}$$

—
$$\begin{aligned} g : \mathbb{R} &\longrightarrow \mathbb{R}^+ \\ x &\longmapsto x^2. \end{aligned}$$

—
$$\begin{aligned} Id_E : E &\longrightarrow E \\ x &\longmapsto x. \end{aligned}$$

Id_E is called the identity function on E.

2. The correspondence :

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \frac{1}{x}. \end{aligned}$$

is not a function, since the element 0 has no image in $\mathbb{R}$.

1.2.1 Direct Image and Inverse Image

a) Direct Image

Definition 1.12. *Let $f : E \to F$ and $A \subset E$. The direct image of A by f, written $f(A)$, is the set of all images of the elements of A by f :*

$$f(A) = \{f(x) \in F \mid x \in A\}.$$

Example 1.12. *Let f be the function defined by :*

$$\begin{aligned} f : [0,3] &\longrightarrow [0,5] \\ x &\longmapsto f(x) = 2x+1 \end{aligned}$$

— *Let's calculate $f(\{2\})$:*

$$f(\{2\}) = \{f(x) \mid x \in \{2\}\} = \{2x+1 \mid x = 2\} = \{f(2)\} = \{5\}.$$

— *Let's calculate $f([0,1])$:*

$f([0,1]) = \{f(x) \mid x \in [0,1]\} = \{2x+1 \mid 0 \le x \le 1\}$.
Since $0 \le x \le 1 \Rightarrow 0 \le 2x \le 2 \Rightarrow 1 \le 2x+1 \le 3$,
then $f([0,1]) = [1,3] \subset [0,5]$.

b) Inverse Image

Definition 1.13. *Let $f : E \to F$ and $B \subset F$. The inverse image of B by f, written $f^{-1}(B)$, is the set of all x in E such that $f(x)$ is in B :*

$$f^{-1}(B) = \{x \in E \mid f(x) \in B\}.$$

Example 1.13. *Let f be the function defined by :*

$$\begin{aligned} f : [0,2] &\longrightarrow [0,4] \\ x &\longmapsto f(x) = (x-1)^2 \end{aligned}$$

1. *Let's find $f^{-1}(\{0\})$:*

   $f^{-1}(\{0\}) = \{x \in [0,2] \mid f(x) \in \{0\}\} = \{x \in [0,2] \mid f(x) = 0\} = \{x \in [0,2] \mid (x-1)^2 = 0\} = \{1\}.$

2. *Let's find $f^{-1}((0,1))$:*

   $$\begin{aligned} f^{-1}((0,1)) &= \{x \in [0,2] \mid f(x) \in (0,1)\} \\ &= \{x \in [0,2] \mid 0 < (x-1)^2 < 1\}. \end{aligned}$$

   We have $(x-1)^2 > 0$ for all $x \in [0,1) \cup (1,2]$.
   Also, $(x-1)^2 < 1 \Rightarrow |x-1| < 1 \Rightarrow -1 < x-1 < 1 \Rightarrow 0 < x < 2$.
   Combining both, we get :

   $$f^{-1}((0,1)) = ([0,1) \cup (1,2]) \cap (0,2) = (0,1) \cup (1,2).$$

Proposition 1.1. *Let $f : E \to F$. Then :*

1. *If $A \subset B$, then $f(A) \subset f(B)$.*
2. *$f(A \cup B) = f(A) \cup f(B)$.*
3. *$f(A \cap B) \subset f(A) \cap f(B)$.*

Démonstration. 1. Let $y \in f(A)$. Then there exists $x \in A$ such that $f(x) = y$. Since $A \subset B$, we have $x \in B$, hence $y = f(x) \in f(B)$. Therefore, $f(A) \subset f(B)$.

2. Let's show that $f(A \cup B) \subset f(A) \cup f(B)$.

   $$\begin{aligned} y \in f(A \cup B) &\Leftrightarrow \exists x \in A \cup B \text{ such that } f(x) = y \\ &\Leftrightarrow (\exists x \in A, f(x) = y) \vee (\exists x \in B, f(x) = y) \\ &\Leftrightarrow y \in f(A) \vee y \in f(B) \\ &\Leftrightarrow y \in f(A) \cup f(B). \end{aligned}$$

   The reverse inclusion $f(A) \cup f(B) \subset f(A \cup B)$ is proved in the same way. Hence, $f(A \cup B) = f(A) \cup f(B)$.

3. Let

   $$\begin{aligned} y \in f(A \cap B) &\Rightarrow \exists x \in A \cap B \text{ such that } f(x) = y \\ &\Rightarrow (\exists x \in A, f(x) = y) \wedge (\exists x \in B, f(x) = y) \\ &\Rightarrow y \in f(A) \wedge y \in f(B) \\ &\Rightarrow y \in f(A) \cap f(B). \end{aligned}$$

   Therefore, $f(A \cap B) \subset f(A) \cap f(B)$.

$\square$

Example 1.14. *Let $f(x) = x^2$, $A = [-1,0]$, $B = [0,1]$. Then $A \cap B = \{0\}$, $f(A) = [0,1]$, and $f(B) = [0,1]$.*

$$f(A) \cap f(B) = [0,1], \quad f(A \cap B) = f(\{0\}) = \{0\} \neq [0,1] = f(A) \cap f(B).$$

The equality $f(A \cap B) = f(A) \cap f(B)$ holds only when f is injective.

1.2.2 Injection, Surjection, Bijection

Let E and F be two non-empty sets, and let f be a function from E to F.

a) Injection

Definition 1.14. *A function f is **injective** (or one-to-one) if each element of F has at most one preimage in E. That is,*

$$f \text{ is injective } \Leftrightarrow \forall x_1, x_2 \in E,\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2,$$

or equivalently,

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

Example 1.15. *Are the following functions injective ?*

1.

$$\begin{aligned} f_1 : \mathbb{N} &\to \mathbb{N} \\ n &\longmapsto 3n + 5. \end{aligned}$$

f_1 is injective because : $\forall n_1, n_2 \in \mathbb{N},\ f_1(n_1) = f_1(n_2) \Rightarrow 3n_1 + 5 = 3n_2 + 5 \Rightarrow 3n_1 = 3n_2 \Rightarrow n_1 = n_2$

2.

$$\begin{aligned} f_2 : \mathbb{R} &\to \mathbb{R}^+ \\ x &\longmapsto x^2. \end{aligned}$$

f_2 is not injective because -1 and 1 have the same image.

b) Surjection

Definition 1.15. *A function f is **surjective** (onto) if every y in F is the image of at least one x in E :*

$$\forall y \in F,\ \exists x \in E \text{ such that } y = f(x).$$

Example 1.16. *Are the following functions surjective ?*

1.

$$\begin{aligned} f_1 : \mathbb{N} &\to \mathbb{N} \\ n &\longmapsto 3n + 5 \end{aligned}$$

f_1 is not surjective. Suppose it is surjective, that is :

$$\forall y \in \mathbb{N}, \exists n \in \mathbb{N} : y = 3n + 5 \Rightarrow n = \frac{y - 5}{3}.$$

But $n = \frac{y-5}{3}$ is not always a natural number, which is a contradiction. Therefore, f_1 is not surjective.

2.

$$\begin{aligned} f_2 : \mathbb{R} &\to \mathbb{R}^+ \\ x &\longmapsto x^2 \end{aligned}$$

f_2 is surjective because every element of $\mathbb{R}^+$ has at least one preimage in $\mathbb{R}$.

c) Bijection

Definition 1.16. *A function f is **bijective** (or a **bijection**) if it is both injective and surjective. That is, every element of F is the image of one and only one element of E :*

$$f \text{ is bijective } \Leftrightarrow \forall y \in F,\ \exists!\, x \in E : y = f(x).$$

Example 1.17.

1. *f_1 is not bijective because it is not surjective.*
2. *f_2 is not bijective because it is not injective.*
3. *Let*

$$\begin{aligned} f_3 : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto 2x + 1 \end{aligned}$$

f_3 is bijective because :
*— it is **injective** : if $f_3(x_1) = f_3(x_2)$, then $2x_1 + 1 = 2x_2 + 1$, so $x_1 = x_2$;*
*— it is **surjective** : for every $y \in \mathbb{R}$, we have $y = 2x + 1$, which gives $x = \frac{y-1}{2} \in \mathbb{R}$.*

Remark 1.4. *If f is bijective, then there exists an inverse function f^{-1} from F to E, and*

$$(f^{-1})^{-1} = f.$$

Example 1.18. *The function f_3 is bijective, and its inverse is defined by :*

$$\begin{aligned} f_3^{-1} : \mathbb{R} &\longrightarrow \mathbb{R} \\ y &\longmapsto \frac{y-1}{2}. \end{aligned}$$

1.2.3 Composition of Functions

Definition 1.17. *Let E, F, and G be sets. If $f : E \to F$ and $g : F \to G$, then the composition of f and g, written $g \circ f$, is the function from E to G defined by*

$$(g \circ f)(x) = g(f(x)).$$

Example 1.19. *We consider the following functions :*

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^2. \end{aligned} \qquad \begin{aligned} g : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x + 1. \end{aligned}$$

Alors :

$$\begin{aligned} g \circ f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^2 + 1. \end{aligned}$$

et

$$\begin{aligned} f \circ g : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto (x + 1)^2. \end{aligned}$$

It is clear that : $g \circ f \neq f \circ g$.

Proposition 1.2.

1. *If f and g are injective, then $g \circ f$ is injective.*
2. *If f and g are surjective, then $g \circ f$ is surjective.*

Démonstration. 1. Suppose that f and g are injective. Let us show that $g \circ f$ is injective. Let $x_1, x_2 \in E$ such that

$$g \circ f(x_1) = g \circ f(x_2).$$

Then

$$g(f(x_1)) = g(f(x_2)).$$

Since g is injective, we have

$$f(x_1) = f(x_2).$$

As f is injective, it follows that

$$x_1 = x_2.$$

Hence, $g \circ f$ is injective.

2. Suppose that f and g are surjective, that is, $f(E) = F$ and $g(F) = G$. Let us show that $g \circ f$ is surjective. We have

$$g \circ f(E) = g(f(E)) = g(F) = G,$$

by the surjectivity of f and g. Therefore, $g \circ f$ is surjective.

□

Remark 1.5. *It follows that the composition of two bijections is a bijection. In particular, for $f : E \to F$ and its inverse $f^{-1} : F \to E$, we have*

$$f^{-1} \circ f = Id_E \quad \textit{and} \quad f \circ f^{-1} = Id_F.$$

Proposition 1.3. *Let $f : E \longrightarrow F$ and $g : F \longrightarrow G$. Then :*

1. *If $g \circ f$ is injective, then f is injective.*
2. *If $g \circ f$ is surjective, then g is surjective.*
3. *If $g \circ f$ is bijective, then f is injective and g is surjective.*

Démonstration. 1. Let $x_1, x_2 \in E$ such that $f(x_1) = f(x_2)$. Then

$$g(f(x_1)) = g(f(x_2)).$$

Since $g \circ f$ is injective, we have $x_1 = x_2$. Hence, f is injective.

2. We have $f(E) \subset F \Rightarrow g \circ f(E) \subset g(F) \subset G$. Since $g \circ f$ is surjective, we get $g \circ f(E) = G$, which implies $G \subset g(F)$. Therefore, $G = g(F)$, and thus g is surjective.

□

1.3 Binary Relations

Definition 1.18. *A relation from a set E to a set F is a correspondence $\mathcal{R}$ that links elements of E with elements of F.*
If x is related to y by $\mathcal{R}$, we write $x\mathcal{R}y$.
If $E = F$, the relation $\mathcal{R}$ is called a binary relation.

Example 1.20. 1. *For all $x, y \in \mathbb{N}$, let $x\mathcal{R}y \Leftrightarrow x$ divides y. Then $\mathcal{R}$ is a binary relation.*

2. *For all $x, y \in \mathbb{R}$, let $x\mathcal{S}y \Leftrightarrow x^2 + 1 = y^2 + 1$. Then $\mathcal{S}$ is a binary relation.*
3. *Let $\mathbb{P}(E)$ be the set of all subsets of a set E.*
   For all $A, B \subset \mathbb{P}(E)$, let $A\mathcal{T}B \Leftrightarrow A \subset B$. Then $\mathcal{T}$ is a binary relation.

1.3.1 Properties of Binary Relations

Let $\mathcal{R}$ be a binary relation on a set E, and let $x, y, z \in E$. We say that $\mathcal{R}$ is :

1. **Reflexive :** $\forall x \in E,\ x\mathcal{R}x$.
2. **Symmetric :** $\forall x, y \in E,\ x\mathcal{R}y \Rightarrow y\mathcal{R}x$.
3. **Antisymmetric :** $\forall x, y \in E,\ (x\mathcal{R}y) \wedge (y\mathcal{R}x) \Rightarrow (x = y)$.
4. **Transitive :** $\forall x, y, z \in E,\ (x\mathcal{R}y) \wedge (y\mathcal{R}z) \Rightarrow (x\mathcal{R}z)$.

1.3.2 Equivalence Relation

Definition 1.19. *A binary relation $\mathcal{R}$ on a set E is an **equivalence relation** if it is reflexive, symmetric, and transitive.*

Example 1.21. *Let $\mathcal{R}$ be a binary relation on $\mathbb{R}$ defined by :*

$$\forall x, y \in \mathbb{R},\ x\mathcal{R}y \Leftrightarrow x + 1 = y + 1.$$

Let us show that $\mathcal{R}$ is an equivalence relation :

a) *$\mathcal{R}$ is reflexive : $\forall x \in \mathbb{R},\ x\mathcal{R}x$.*
Indeed, $x\mathcal{R}x \Leftrightarrow x + 1 = x + 1 \Rightarrow 0 = 0$, which is true.
Therefore, $\mathcal{R}$ is reflexive.

b) *$\mathcal{R}$ is symmetric : $\forall x, y \in \mathbb{R},\ x\mathcal{R}y \Rightarrow y\mathcal{R}x$.*
We have $x\mathcal{R}y \Leftrightarrow x + 1 = y + 1 \Leftrightarrow y + 1 = x + 1 \Rightarrow y\mathcal{R}x$.
Therefore, $\mathcal{R}$ is symmetric.

c) *$\mathcal{R}$ is transitive : $\forall x, y, z \in \mathbb{R}$,*

$$\begin{cases} x\mathcal{R}y, \\ y\mathcal{R}z \end{cases} \Rightarrow x\mathcal{R}z.$$

We have

$$\begin{cases} x\mathcal{R}y \Leftrightarrow x + 1 = y + 1, \\ y\mathcal{R}z \Leftrightarrow y + 1 = z + 1, \end{cases} \Rightarrow x + 1 = z + 1 \Leftrightarrow x\mathcal{R}z.$$

Therefore, $\mathcal{R}$ is transitive.

*Hence, from **(a)**, **(b)**, and **(c)**, $\mathcal{R}$ is an equivalence relation.*

Equivalence Class

Definition 1.20. *Let $\mathcal{R}$ be an equivalence relation on a set E. The **equivalence class** of $x \in E$ is the set*

$$\bar{x} = \{\, y \in E \mid x\mathcal{R}y \,\}.$$

Remark 1.6. *The set of all equivalence classes of elements of E is called the **quotient set** of E by $\mathcal{R}$, and is denoted by $E_{/\mathcal{R}}$:*

$$E_{/\mathcal{R}} = \{\, \dot{x} \mid x \in E \,\}.$$

Example 1.22. *Let $\mathcal{R}$ be the binary relation on $\mathbb{R}$ defined by :*

$$\forall x, y \in \mathbb{R},\ x\mathcal{R}y \Leftrightarrow x^2 - x = y^2 - y.$$

$\mathcal{R}$ is an equivalence relation because :

1. $\forall x \in \mathbb{R},\ x^2 - x = x^2 - x \Leftrightarrow x\mathcal{R}x$. *Hence, $\mathcal{R}$ is reflexive.*
2. $\forall x, y \in \mathbb{R},\ x\mathcal{R}y \Leftrightarrow x^2 - x = y^2 - y \Leftrightarrow y^2 - y = x^2 - x \Leftrightarrow y\mathcal{R}x$. *Hence, $\mathcal{R}$ is symmetric.*
3. $\forall x, y, z \in \mathbb{R},\ (x\mathcal{R}y \wedge y\mathcal{R}z) \Leftrightarrow (x^2 - x = y^2 - y \wedge y^2 - y = z^2 - z) \Rightarrow x^2 - x = z^2 - z \Leftrightarrow x\mathcal{R}z$. *Hence, $\mathcal{R}$ is transitive.*

Now, let us find the following equivalence classes : C_0, $\overline{1}$, $\dot{2}$, *and* $C_{\frac{1}{2}}$.

1. $C_0 = \{y \in E \mid 0\mathcal{R}y\}$, *and* $0\mathcal{R}y \Leftrightarrow y^2 - y = 0$, *hence* $C_0 = \{0, 1\}$.
2. $\overline{1} = \{y \in E \mid 1\mathcal{R}y\}$, *and* $y^2 - y = 1 - 1 = 0$, *hence* $\overline{1} = \{0, 1\}$.
3. $\dot{2} = \{y \in E \mid 2\mathcal{R}y\}$, *and* $y^2 - y = 2$, *hence* $\dot{2} = \{-1, 2\}$.
4. $C_{\frac{1}{2}} = \{y \in E \mid \frac{1}{2}\mathcal{R}y\}$, *and* $y^2 - y = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$, *hence* $C_{\frac{1}{2}} = \left\{\frac{1}{2}\right\}$.

1.3.3 Order Relation

Definition 1.21. *Let E be a non-empty set. A binary relation $\mathcal{R}$ on E is called an **order relation** if it is reflexive, antisymmetric, and transitive.*

Example 1.23. *Let $\mathcal{R}$ be the binary relation on E defined by :*

$$\forall A \subset E, \forall B \subset E, \quad A\mathcal{R}B \Leftrightarrow A \subset B.$$

Let us show that $\mathcal{R}$ is an order relation :

a) *Reflexive :* $\forall A \subset E,\ A \subset A$. *Hence, $\mathcal{R}$ is reflexive.*

b) *Antisymmetric :* $\forall A, B \subset E,\ (A \subset B \wedge B \subset A) \Rightarrow A = B$. *Hence, $\mathcal{R}$ is antisymmetric.*

c) *Transitive :* $\forall A, B, C \subset E,\ (A \subset B \wedge B \subset C) \Rightarrow A \subset C$. *Hence, $\mathcal{R}$ is transitive.*

*Therefore, from **(a)**, **(b)**, and **(c)**, $\mathcal{R}$ is an order relation.*

Definition 1.22. *An order relation on a set E is called a **total order** if any two elements of E are comparable, i.e., $\forall x, y \in E$, we have $x\mathcal{R}y$ or $y\mathcal{R}x$. An order relation that is not total is called a **partial order**.*

Example 1.24.
1. *For all $x, y \in \mathbb{R}$, let $x\mathcal{R}y \Leftrightarrow x \leq y$. This is a total order relation because :*
2. *Reflexive :* $\forall x \in \mathbb{R},\ x \leq x \Leftrightarrow x\mathcal{R}x$.
3. *Antisymmetric :* $\forall x, y \in \mathbb{R},\ (x\mathcal{R}y \wedge y\mathcal{R}x) \Leftrightarrow (x \leq y \wedge y \leq x) \Rightarrow x = y$.
4. *Transitive :* $\forall x, y, z \in \mathbb{R},\ (x\mathcal{R}y \wedge y\mathcal{R}z) \Leftrightarrow (x \leq y \wedge y \leq z) \Rightarrow x \leq z \Leftrightarrow x\mathcal{R}z$.
5. *Total :* $\forall x, y \in \mathbb{R},\ x \leq y$ *or* $y \leq x$, *so $\mathcal{R}$ is a total order relation.*