I.2.1. Scalar quantities and vector quantities المقادير السلمية و المقادير الشعاعية

Physical quantities are divided into two groups:

- Scalar quantity such that; mass (m), time (t), energy (E), ...

- Vector quantity such as velocity ($\vec{v}$), force ($\vec{F}$),...

I.2.2. Vectors الأشعة

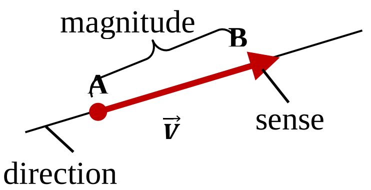
I.2.2.1. Definition: تعريف

A vector is a line segment AB, having an origin A and an end B. We denote it by $\overrightarrow{AB}$, or by a single letter: $\overrightarrow{AB} = \vec{V}$.characterized by:

- Its direction which is defined by that of the line which carries the segment

- Its sense which designates the orientation of the vector (from A towards B).

- Its magnitude (norm or intensity) which is equal to the length of the segment [AB], noted $\|\overrightarrow{AB}\|$ which is always positive.

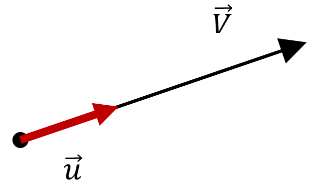


I.2.2.2. Unit vector شعاع الوحدة

A vector is unitary when its magnitude is equal to unity (1).

If $\vec{u}$ is a unit vector carried by a vector $\vec{V}$ then:

$$\vec{V} = \|\vec{V}\|.\vec{u} \Rightarrow \vec{u} = \frac{\vec{V}}{\|\vec{V}\|}$$



We also have $\|\vec{u}\| = 1$ and $\vec{u}$ is always parallel to $\vec{V}$($\vec{u}// \vec{V}$)

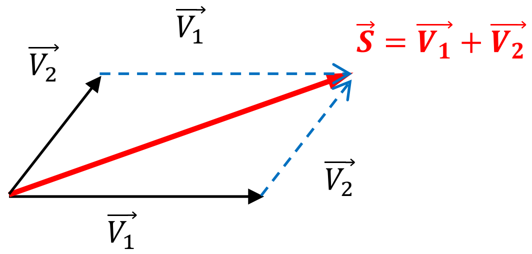
I.2.3. Vector Operations

Let $\vec{V_1}$, $\vec{V_2}$, $\vec{V_3}$, be three vectors, *a, b and c* real numbers

I.2.3.1. The sum (addition) of the vectors جمع الأشعة

The sum of two vectors $\vec{V_1}$ and $\vec{V_2}$ is $\quad \vec{S} = \vec{V_1} + \vec{V_2}$

Graphically, we can find the resulting vector $\vec{S}$ by the parallelogram rule.



The sum of n vectors: $\vec{V_1}, \vec{V_2}, \vec{V_3},\ldots \vec{V_n}$. is $\quad \vec{S} = \vec{V_1} + \vec{V_2} + \vec{V_3} + \cdots \vec{V_n}$

Properties: الخواص

* Commutativity (تبديلي): $\vec{S} = \vec{V_1} + \vec{V_2} = \vec{V_2} + \vec{V_1}$

* Associativity (تجميعي): $(\vec{V_1} + \vec{V_2}) + \vec{V_3} = \vec{V_1} + (\vec{V_2} + \vec{V_3})$.

* Distributivity (توزيعي) : $(a+b).\ \vec{V_1} = a.\ \vec{V_1} + b.\ \vec{V_1}$ and $a.(\ \vec{V_1} + \vec{V_2}) = a.\ \vec{V_1} + a.\ \vec{V_2}$

* The sum of a vector $\vec{V_1}$ and its opposite (- $\vec{V_1}$) is zero: $\vec{V_1} + (-\vec{V_1}) = \vec{0}$

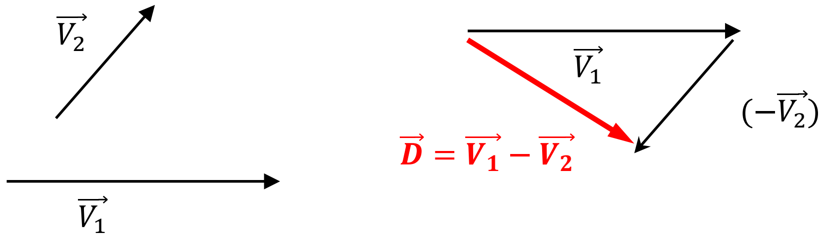
I.2.3.2. Vector subtraction طرح الأشعة

The difference of two vectors $\vec{V_1}$ and $\vec{V_2}$ is a vector $\vec{D}$, with :

$$\vec{D} = \vec{V_1} - \vec{V_2} = \vec{V_1} + (-\vec{V_2}) \neq \vec{V_2} - \vec{V_1}$$

The difference of the vectors is non-commutative.

Graphically, we can find the resulting vector $\vec{D}$ by the parallelogram rule.



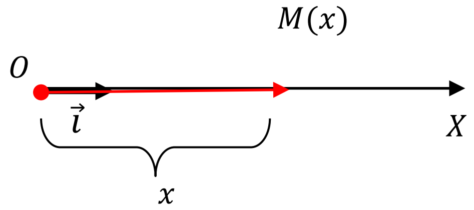
I.2.3.3. Components of a vector مركبات شعاع

To determine the components of a vector, it is necessary to choose a reference frame (coordinate system) which is a set of non-collinear unit vectors called basis. We can then decompose all the other vectors according to these unit vectors and this decomposition is unique. We have three types of references frame:

a) Linear reference frame: معلم خطي

It is composed of a single axis Ox, provided with a unit vector $\vec{\imath}$ positively oriented. The coordinate (x) of point M is defined by: $\overrightarrow{OM} = x\vec{\imath}$

(x) is also called the component of the vector $\overrightarrow{OM}$.



b) Planar (two-dimensional) orthogonal reference frame: معلم مستوي

It is composed of two orthogonal axes of the plane, OX and OY, provided with unit vectors $\vec{\imath}$ *and* $\vec{\jmath}$ positively oriented.

The position of a point M is characterized by the vector $\overrightarrow{OM}$: $\vec{V} = \overrightarrow{OM}$

Let x and y be the projections of M onto the OX and OY axes respectively,we have

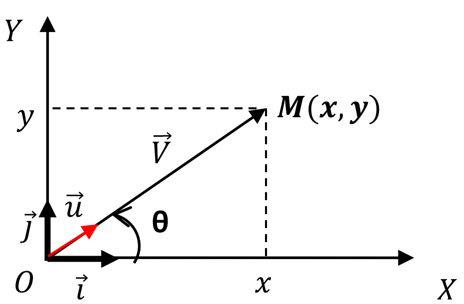
$$\vec{V} = x\vec{\imath} + y\vec{\jmath} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{So} \quad \begin{cases} x = \|\vec{V}\|cos\theta \\ y = \|\vec{V}\|sin\theta \end{cases}$$

$$\vec{V} = x\vec{\imath} + y\vec{\jmath} = \|\vec{V}\|cos\theta\vec{\imath} + \|\vec{V}\|sin\theta\vec{\jmath}\vec{u}$$

$$\Rightarrow \vec{V} = \|\vec{V}\|(\underbrace{cos\theta.\vec{\imath} + sin\theta.\vec{\jmath}}_{\vec{u}}) \Rightarrow \vec{V} = \|V\|.\vec{u}$$

$\vec{u}$ is the unit vector of the vector $\vec{V}$:

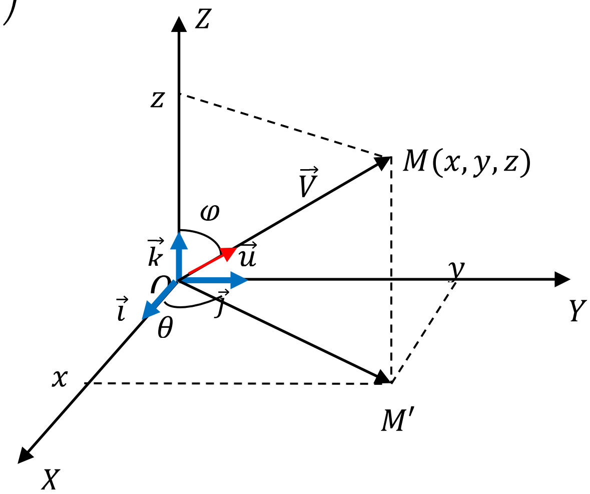
$$\vec{u} = cos\theta.\vec{\imath} + sin\theta.\vec{\jmath}$$



(x,y) is called the components of the vector $\vec{V}$ or the cartesian coordinates of the point M in the plane (OXY)

c) An orthonormal reference in space: معلم متعامد متجانس في الفضاء

It is composed of three orthogonal axes, OX, OY and OZ, provided with unit vectors $\vec{i}, \vec{j}$ *and* $\vec{k}$ positively oriented. The position of a point M in space is characterized by the vector $\vec{V} = \overrightarrow{OM}$. Let x , y and z be the projections of M onto the axes OX, OY and OZ, respectively. So we have: $\vec{V} = x\vec{i} + y\vec{j} + z\vec{k} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$



$$\begin{cases} x = \left\|\overrightarrow{OM'}\right\| \cos\theta \\ y = \left\|\overrightarrow{OM'}\right\| \sin\theta \Rightarrow \left\|\overrightarrow{OM'}\right\| = \|\vec{V}\|.\sin\varphi \\ z = \|\vec{V}\|\cos\varphi \end{cases}$$

$$\Rightarrow \begin{cases} x = \|\vec{V}\|\sin\varphi.\cos\theta \\ y = \|\vec{V}\|\sin\varphi.\sin\theta \\ z = \|\vec{V}\|\cos\varphi \end{cases}$$

$\vec{V} = \|\vec{V}\|.\vec{u}$

$\vec{u}$ is the unit vector of the vector $\vec{V}$

$\vec{u} = \sin\varphi.\cos\theta.\vec{i} + \sin\varphi.\sin\theta.\vec{j} + \cos\varphi.\vec{k}$

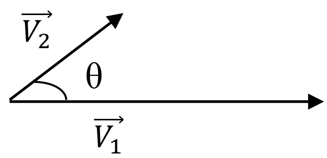
(x,y,z) is called the components of the vector $\overrightarrow{OM}$ or the cartesian coordinates of the point M in the orthonormal reference frame (OXYZ).

I.2.3.4. Magnitude (norm) of a vector طويلة شعاع

The magnitude of a vector $\vec{V} = x\vec{i} + y\vec{j} + z\vec{k}$ represents its length, it is given by the following formula: $\|\vec{V}\| = \sqrt{x^2 + y^2 + z^2} = V$. $\|\vec{V}\|$ is always positive

I.2.4. Scalar (dot) product الجداء السلمي

The dot product of two vectors $\vec{V_1}$ and $\vec{V_2}$ is a scalar given by the following relation:



$$\vec{V_1}.\vec{V_2} = \|\vec{V_1}\|.\|\vec{V_2}\|.\cos\theta$$

Where θ is the angle between the two vectors $\vec{V_1}$and $\vec{V_2}$

Properties

- $\vec{V}.\vec{V} = \|\vec{V}\|.\|\vec{V}\|.\cos 0 = V^2$
- $\vec{V_1}.\vec{V_2} = \|\vec{V_1}\|.\|\vec{V_2}\|.\cos\theta = \|\vec{V_2}\|.\|\vec{V_1}\|.\cos(-\theta) \Rightarrow \vec{V_1}.\vec{V_2} = \vec{V_2}.\vec{V_1}$
- $\vec{V_1}.(\vec{V_2} + \vec{V_3}) = \vec{V_1}.\vec{V_2} + \vec{V_1}.\vec{V_3}$
- $(\vec{V_1} \pm \vec{V_2})^2 = {V_1}^2 + {V_2}^2 \pm 2V_1V_2\cos\theta$
- If $\theta = \frac{\pi}{2}$, their scalar product is zero: $\vec{V_1} \perp \vec{V_2} \Rightarrow \vec{V_1}.\vec{V_2} = 0$
- $\vec{i}.\vec{i} = \vec{j}.\vec{j} = \vec{k}.\vec{k} = 1$ and $\vec{i}.\vec{j} = \vec{j}.\vec{k} = \vec{i}.\vec{k} = 0$

❖ If we know the coordinates of two vectors in an orthonormal basis, the scalar product will be expressed only in terms of the coordinates:

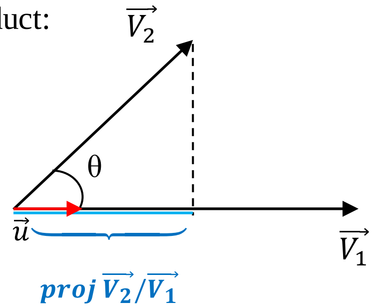
$$\overrightarrow{V_1}\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix} \text{ and } \overrightarrow{V_2}\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix} \Rightarrow \overrightarrow{V_1}.\overrightarrow{V_2} = \begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}.\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2$$

I.2.5. Projection of the vector: مسقط شعاع

The projection of the vector $\overrightarrow{V_2}$ onto $\overrightarrow{V_1}$ is given by the following relation:

$$proj\,\overrightarrow{V_2}/\overrightarrow{V_1} = \|\overrightarrow{V_2}\|.\cos\theta$$

We can rewrite the previous relation in the form of a scalar product:



$\vec{u}.\overrightarrow{V_2} = \|\vec{u}\|.\|\overrightarrow{V_2}\|.\cos\theta = \|\vec{u}\|.proj\,\overrightarrow{V_2}/\overrightarrow{V_1}$

$\vec{u}$ is the unit vector of the vector $\overrightarrow{V_1}$ $\quad \Rightarrow \|\vec{u}\| = \frac{\overrightarrow{V_1}}{\|\overrightarrow{V_1}\|} = 1$

$$\Rightarrow proj\,\overrightarrow{V_2}/\overrightarrow{V_1} = \frac{\overrightarrow{V_1}.\overrightarrow{V_2}}{V_1}$$

I.2.5.1. Vector projection of vector: شعاع مسقط شعاع

The vector projection of vector $\overrightarrow{V_2}$ onto $\overrightarrow{V_1}$ is a vector defined by:

$$\overrightarrow{proj}\,\overrightarrow{V_2}/\overrightarrow{V_1} = proj\frac{\overrightarrow{V_2}}{\overrightarrow{V_1}}.\vec{u} = \frac{\overrightarrow{V_1}.\overrightarrow{V_2}}{V_1}.\vec{u} = \frac{(\overrightarrow{V_1}.\overrightarrow{V_2})}{V_1}.\frac{\overrightarrow{V_1}}{V_1} = \frac{\overrightarrow{V_1}.(\overrightarrow{V_1}.\overrightarrow{V_2})}{{V_1}^2}$$

$$\overrightarrow{proj}\,\overrightarrow{V_2}/\overrightarrow{V_1} = \frac{\overrightarrow{V_1}.(\overrightarrow{V_1}.\overrightarrow{V_2})}{{V_1}^2}$$

I.2.5.2. Direction cosines

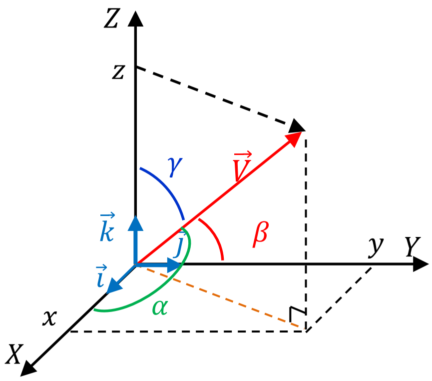
The direction cosines of the vector $\vec{V} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ are the cosines of angles that the vector $\vec{V}$ forms with the coordinate axes.

Let α, β and γ be the angles that the vector $\vec{V}$ makes with the axes OX, OY and OZ.

$\cos\alpha = \frac{x}{V}$

$\cos\beta = \frac{y}{V}$

$\cos\gamma = \frac{z}{V}$



We have: $\vec{V} = \|\vec{V}\|.\vec{u} \Rightarrow \vec{u} = \frac{\vec{V}}{\|\vec{V}\|}$

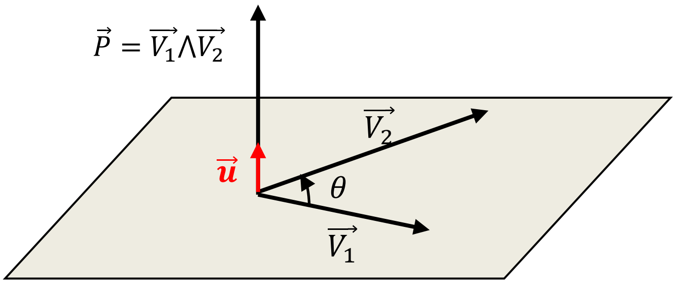
$$\vec{u} = \frac{x}{\|\vec{V}\|}\vec{\imath} + \frac{y}{\|\vec{V}\|}\vec{\jmath} + \frac{z}{\|\vec{V}\|}\vec{k}$$

$$\vec{u} = \cos\alpha.\vec{\imath} + \cos\beta.\vec{\jmath} + \cos\gamma.\vec{k}$$

The direction cosines of the vector $\vec{V}$ are the components of the unit vector $\vec{u}$ of the vector $\vec{V}$

I.2.6. vector (cross) product الجداء الشعاعي

The cross product of two vectors $\vec{V_1}$ and $\vec{V_2}$ is another vector $\vec{P}$ perpendicular to the plane which formed by two vectors, it's direction is found by using the right-hand rule. The vector product is defined by:

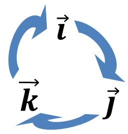


$$\vec{P} = \vec{V_1} \wedge \vec{V_2} = \|\vec{V_1}\| . \|\vec{V_2}\| . sin\theta . \vec{u}$$

Where $\vec{u}$ is the unit vector perpendicular to plane formed by $\vec{V_1}$ and $\vec{V_2}$.

Properties

- ❖ The magnitude of $\vec{P}$ is given by :$\| \vec{P}\| = \|\vec{V_1} \wedge \vec{V_2}\| = \|\vec{V_1}\| . \|\vec{V_2}\| . |sin\theta|$
- ❖ The cross product is anticommutative: $\vec{V_1} \wedge \vec{V_2} = -(\vec{V_2} \wedge \vec{V_1})$
- ❖ The cross product is distributive: $\vec{V_1} \wedge (\vec{V_2} \pm \vec{V_3}) = \vec{V_1} \wedge \vec{V_2} \pm \vec{V_1} \wedge \vec{V_3}$
- ❖ $\vec{V_1} \wedge \vec{V_2} = \vec{0} \Rightarrow \vec{V_1} \ // \ \vec{V_2}$
- ❖ $\vec{\imath} \wedge \vec{\imath} = \vec{\jmath} \wedge \vec{\jmath} = \vec{k} \wedge \vec{k} = \vec{0}$ And $\vec{\imath} \wedge \vec{\jmath} = \vec{k}, \ \vec{\jmath} \wedge \vec{k} = \vec{\imath}, \ \vec{k} \wedge \vec{\imath} = \vec{\jmath}$



- ❖ The cross product can be calculated by the determinant method

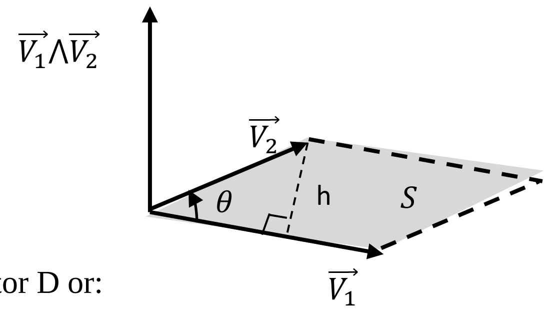
based on the coordinates of $\vec{V_1}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\vec{V_2}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$:

$$\vec{V_1} \wedge \vec{V_2} = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} . \vec{\imath} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} . \vec{\jmath} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} . \vec{k}$$

$$\vec{V_1} \wedge \vec{V_2} = (y_1 z_2 - z_1 y_2) . \vec{\imath} - (x_1 z_2 - z_1 x_2) . \vec{\jmath} + (x_1 y_2 - y_1 x_2) . \vec{k}$$

I.2.6.1. Magnitude of the cross product طويلة الجداء الشعاعي

The magnitude of the cross product of two vectors represents the area of a parallelogram formed by these two vectors:



$$\|\vec{V_1} \wedge \vec{V_2}\| = \|\vec{V_1}\| . \|\vec{V_2}\| . |sin\theta|$$

$$h = V_2 . |sin\theta| \Rightarrow S = h . V_1 = \|\vec{V_1} \wedge \vec{V_2}\|$$

I.2.6.2. Triple product: الجداء المضاعف

The triple product of $\vec{V_1}, \vec{V_2}$ *and* $\vec{V_3}$ is defined by the vector D or:

$$\vec{D} = \vec{V_1} \wedge (\vec{V_2} \wedge \vec{V_3}) = (\vec{V_1} . \vec{V_3}) . \vec{V_2} - (\vec{V_1} . \vec{V_2}) . \vec{V_3}$$

I.2.6.2. Mixed product الجداء المختلط

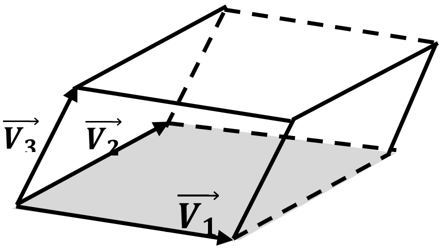
The mixed product of three vectors $\vec{V_1}, \vec{V_2}$ *and* $\vec{V_3}$ is the scalar quantity defined by

$$\vec{V_1}.\left(\vec{V_2}\wedge\vec{V_3}\right)=\begin{vmatrix} x_1 & -y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\end{vmatrix}=\begin{vmatrix} y_2 & z_2\\ y_3 & z_3\end{vmatrix}.x_1-\begin{vmatrix} x_2 & z_2\\ x_3 & z_3\end{vmatrix}.y_1+\begin{vmatrix} x_2 & y_2\\ x_3 & y_3\end{vmatrix}.z_1$$

$$\vec{V_1}.\left(\vec{V_2}\wedge\vec{V_3}\right)=(y_2.z_3-z_2.y_3).x_1-(x_2.z_3-z_2.x_3).y_1+(x_2.y_3-y_2.x_3).z_1$$

Note :

The value obtained from the mixed product of the three vectors is equal to the volume of the parallelepipedformed by these three vectors.



Properties:

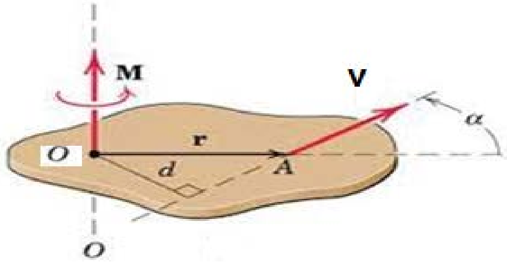
- $\vec{V_1}.\left(\vec{V_2}\wedge\vec{V_3}\right)=\vec{V_3}.\left(\vec{V_1}\wedge\vec{V_2}\right)=\vec{V_2}.\left(\vec{V_3}\wedge\vec{V_1}\right)$
- $\vec{V_1}.\left(\vec{V_2}\wedge\vec{V_3}\right)=0$ ⇒Either the three vectors are in the same plane or $\vec{V_2}\parallel\vec{V_3}$.

I.2.7. Moment of a vector عزم شعاع

I.2.7.1. Moment of a vector relative to a point عزم شعاع بالنسبة إلى نقطة

The moment of a vector $\vec{V_1}$, which passes through point A, relative to a point O is defined by the vector $\vec{\mathcal{M}}$ such that:

$\vec{\mathcal{M}}_{\vec{V_1}/O}=\vec{OA}\wedge\vec{V_1}$



I.2.7.2. Moment of a vector relative to an axis عزم شعاع بالنسبة إلى محور

The moment of a vector $\vec{V_1}$, which passes through point A, relative to an axis(Δ) is given by the scalar product $\mathcal{M}$ such that:

$$\mathcal{M}_{\vec{V_1}/(\Delta)}=\vec{\mathcal{M}}_{\vec{V_1}/O}=\left(\vec{OA}\wedge\vec{V_1}\right).\vec{u_\Delta}$$

$\vec{u_\Delta}$: the unit vector of the axis (Δ).

I.2.8. Vector derivatives

Let a vector $\vec{V}$(vector function)depend on tim e (t): $\vec{V}(t)=x(t)\vec{\imath}+y(t)\vec{\jmath}+z(t)\vec{k}$

$$\frac{d\vec{V}}{dt}=\frac{dx}{dt}\vec{i}+\frac{dy}{dt}\vec{j}+\frac{dz}{dt}\vec{k}$$

I.2.8.1. Properties

Consider two vector functions $\vec{A}$(t) and $\vec{B}$(t) and f(t) a scalar function:

- $\frac{d}{dt}\left(\vec{A}+\vec{B}\right)=\frac{d\vec{A}}{dt}+\frac{d\vec{B}}{dt}$
- $\frac{d}{dt}\left(f.\vec{A}\right)=\frac{df}{dt}+f.\frac{d\vec{A}}{dt}$
- $\frac{d}{dt}\left(\vec{A}.\vec{B}\right)=\frac{d\vec{A}}{dt}.\vec{B}+\vec{A}.\frac{d\vec{B}}{dt}$

- $\frac{d}{dt}(\vec{A}\wedge\vec{B}) = \frac{d\vec{A}}{dt}\wedge\vec{B} + \vec{A}\wedge\frac{d\vec{B}}{dt}$

I.2.8.2. Vector analysis التحليل الشعاعي

a) "Nabla" operator المؤثر نبلا

The nabla operator $\vec{\nabla}$ is a vector quantity written in cartesian coordinates:

$$\vec{\nabla}= \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

b) "Gradient" operator مؤثر التدرج

Let *f(x, y, z)* be a scalar function. Gradient of *f* is given by the following vector:

$$\overrightarrow{\text{grad}}\, f = \vec{\nabla} f = \left(\frac{\partial f}{\partial x}\right)\vec{i} + \left(\frac{\partial f}{\partial y}\right)\vec{j} + \left(\frac{\partial f}{\partial z}\right)\vec{k}$$

c) "Divergence" operator مثر التباعد

Let it be $\vec{V} = V_x\vec{i} + V_y\vec{j} + V_z\vec{k}$ a vector function. We define $div\vec{V}$ as follows:

$$\text{div}\vec{V} = \vec{\nabla}.\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

d) "Curl" operator مؤثر الدوران

Let it be $\vec{V} = V_x\vec{i} + V_y\vec{j} + V_z\vec{k}$ a vector function. We define $\overrightarrow{rot}(\vec{V})$ as follows:

$$\overrightarrow{\text{rot}}(\vec{V}) = \vec{\nabla}\wedge\vec{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\vec{i} - \left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right)\vec{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\vec{k}$$

e) "Laplacian" operator مؤثر لابلاسيان

- Laplacian of a scalar function is defined by the following relation:

$$\vec{\nabla}^2.(f) = \vec{\nabla}.\vec{\nabla}(f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- Laplacian of a vector function is given by the following relation:

$$\vec{\nabla}^2.(\vec{V}) = \vec{\nabla}.\vec{\nabla}(\vec{V}) = \frac{\partial^2 V_x}{\partial x^2}\vec{i} + \frac{\partial^2 V_y}{\partial y^2}\vec{j} + \frac{\partial^2 V_z}{\partial z^2}\vec{k}$$

Exercises

Exercise 1

Let the vectors in space be $\vec{V_1} = 2\vec{\imath} - \vec{\jmath} + 3\vec{k}$ and $\vec{V_2} = -\vec{\imath} + \vec{\jmath} + 2\vec{k}$ represented in the frame $R(O, \vec{\imath}, \vec{\jmath}, \vec{k})$

-Calculate the angle between the two vectors $\vec{V_1}$ and $\vec{V_2}$.

Exercise 2

Let the vectors in space be represented in an orthonormal coordinate system $R\ (OXYZ)$,

$\vec{V_1} = 2\vec{\imath} - 3\vec{\jmath} + \vec{K}$ and $\vec{V_2} = -\vec{\imath} + 2\vec{\jmath} + \vec{K}$

1. Represent these vectors in the reference $R(OXYZ)$.
2. Calculate $\vec{R} = \vec{V_1} + \vec{V_2}$ and the modules: $\|\vec{V_1}\|$, $\|\vec{V_2}\|$.
3. Calculate the scalar product of $\vec{V_1}\, and\ \vec{V_2}$ and deduce the angle between them.
4. Determine the unit vector carried by the vector $\vec{V_2}$. Deduce the direction cosines of $\vec{V_2}$.

Exercise 3

Let the vectors in space be represented in an orthonormal coordinate system $R\ (OXYZ)$,

$\vec{V_2} = \vec{\imath} + \vec{\jmath} + \vec{k}$ and $\vec{V_1} = 2\vec{\imath} + \vec{\jmath} - \vec{k}$

-Calculate the projection and the vector projection of the vector $\vec{V_2}$ onto the vector $\vec{V_1}$.

Exercise 4

Consider the points A(1,0,-1), B(-1,2,1), C(2,1,3) and D(0,1,0) in the frame (OXYZ).

1- Determine the components and magnitudes of the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$.

2- Determine the projection and the vector projection of $\overrightarrow{AB}$ on $\overrightarrow{AC}$.

3- Calculate the surface (area) of triangle ABC and the volume constituted by $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$.

Exercise 5

In the frame $R(O, \vec{\imath}, \vec{\jmath}, \vec{k})$ we give the sliding vector $\vec{V} = \vec{\imath} + 2\,\vec{\jmath} + 3\vec{k}$ and which passes through the point A(3, 4, 2).

1. Calculate the moment of the vector $\vec{V}$ relative to the origin O, then relative to the axes OX and OY.
2. Calculate the moment of vector $\vec{V}$ relative to point B (3, 6, 0)
3. Consider the (Δ) axis of unit vector $\vec{u}$ (-1/√2, 1/2, 1/2) and passing through B, calculate the moment of $\vec{V}$ relative to (Δ).

Solution

Exercise 1

We have $\vec{V_1}.\vec{V_2} = \|\vec{V_1}\|.\|\vec{V_2}\|.cos\theta \Rightarrow cos\theta = \frac{\vec{V_1}.\vec{V_2}}{\|\vec{V_1}\|.\|\vec{V_2}\|}$

$$\vec{V_1}.\vec{V_2} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = 2.(-1) + (-1).1 + 3.(2) = -2 - 1 + 6 = 3$$

$$\|\vec{V_1}\| = \sqrt{{x_1}^2 + {y_1}^2 + {z_1}^2} = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{4+1+9} = \sqrt{14}$$

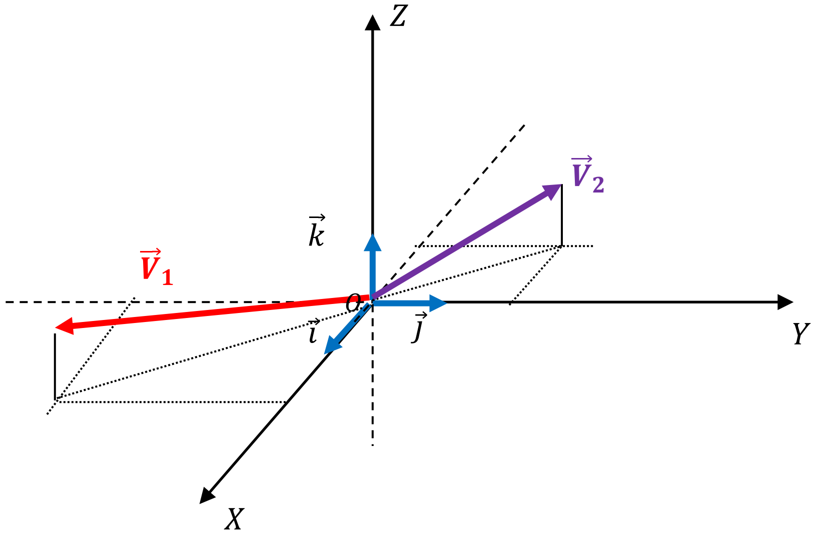
$$\|\vec{V_2}\| = \sqrt{{x_2}^2 + {y_2}^2 + {z_2}^2} = \sqrt{(-1)^2 + 1^2 + 2^2} = \sqrt{1+1+4} = \sqrt{6}$$

$$cos\theta = \frac{\vec{V_1}.\vec{V_2}}{\|\vec{V_1}\|.\|\vec{V_2}\|} = \frac{3}{\sqrt{14}\sqrt{6}} = 0.32$$

$$\Rightarrow \boxed{\theta\ = 71.33}$$

Exercise 2

1. Represent $\vec{V_1} and\ \vec{V_2}$ in the reference $R(OXYZ)$.



2. $$\vec{R} = \vec{V_1} + \vec{V_2} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \vec{\imath} - \vec{\jmath} + 2\vec{K}$$

$$\boxed{\vec{R} = \vec{\imath} - \vec{\jmath} + 2\vec{K}}$$

$$\|\vec{V_1}\| = \sqrt{{x_1}^2 + {y_1}^2 + {z_1}^2} = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{4+9+1} = \sqrt{14}$$

$$\boxed{\|\vec{V_1}\| = \sqrt{14}}$$

$$\|\vec{V_2}\| = \sqrt{{x_2}^2 + {y_2}^2 + {z_2}^2} = \sqrt{(-1)^2 + 2^2 + 1^2} = \sqrt{1+4+1} = \sqrt{6}$$

$$\boxed{\|\vec{V_2}\| = \sqrt{6}}$$

3. We have $\overrightarrow{V_1}.\overrightarrow{V_2} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 2(-1) + (-3)2 + 1(1) = -2 - 6 + 1 = -7$

$$\boxed{\overrightarrow{V_1}.\overrightarrow{V_2} = -7}$$

$\overrightarrow{V_1}.\overrightarrow{V_2} = \|\overrightarrow{V_1}\|.\|\overrightarrow{V_2}\|.cos\theta \Rightarrow cos\theta = \dfrac{\overrightarrow{V_1}.\overrightarrow{V_2}}{\|\overrightarrow{V_1}\|.\|\overrightarrow{V_2}\|}$

$cos\theta = \dfrac{-7}{\sqrt{14}\sqrt{6}} = -0.76$

$$\boxed{\theta = 139.79°}$$

3. unit vector carried by the vector $\overrightarrow{V_2}$:

$\overrightarrow{V_2} = \|\overrightarrow{V_2}\|.\overrightarrow{u_2} \Rightarrow \overrightarrow{u_2} = \dfrac{\overrightarrow{V_2}}{\|\overrightarrow{V_2}\|} = \dfrac{-1}{\sqrt{6}}\vec{\imath} + \dfrac{2}{\sqrt{6}}\vec{\jmath} + \dfrac{1}{\sqrt{6}}\vec{k}$

$$\boxed{\overrightarrow{u_2} = \frac{-1}{\sqrt{6}}\vec{\imath} + \frac{2}{\sqrt{6}}\vec{\jmath} + \frac{1}{\sqrt{6}}\vec{k}}$$

- The direction cosines of $\overrightarrow{V_2}$ are the components of unit vector carried by the vector $\overrightarrow{V_2}$

$$\overrightarrow{u_2} = \frac{-1}{\sqrt{6}}\vec{\imath} + \frac{2}{\sqrt{6}}\vec{\jmath} + \frac{1}{\sqrt{6}}\vec{K} = cos\alpha.\vec{\imath} + cos\beta.\vec{\jmath} + cos\gamma.\vec{K} \Rightarrow \begin{cases} cos\alpha = \dfrac{-1}{\sqrt{6}} \\ cos\beta = \dfrac{2}{\sqrt{6}} \\ cos\gamma = \dfrac{1}{\sqrt{6}} \end{cases}$$

Exercise 3

1) The projection of the vector $\overrightarrow{V_2}$ onto the vector $\overrightarrow{V_1}$:

$$proj_{\overrightarrow{V_2}/\overrightarrow{V_1}} = \frac{\overrightarrow{V_1}.\overrightarrow{V_2}}{V_1}$$

$\overrightarrow{V_1}.\overrightarrow{V_2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 1.2 + 1.1 + 1.(-1) = 2 + 1 - 1 = 2$

$V_1 = \sqrt{{x_1}^2 + {y_1}^2 + {z_1}^2} = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$

$proj_{\overrightarrow{V_2}/\overrightarrow{V_1}} = \dfrac{\overrightarrow{V_1}.\overrightarrow{V_2}}{V_1} = \dfrac{2}{\sqrt{6}}$

$$\boxed{proj_{\overrightarrow{V_2}/\overrightarrow{V_1}} = \frac{2}{\sqrt{6}}}$$

2) the vector projection of the vector $\overrightarrow{V_2}$ onto the vector $\overrightarrow{V_1}$:

$\overrightarrow{proj}_{\overrightarrow{V_2}/\overrightarrow{V_1}} = \dfrac{\overrightarrow{V_1}.(\overrightarrow{V_1}.\overrightarrow{V_2})}{{V_1}^2}$

${V_1}^2 = 6$

$$\Rightarrow \overrightarrow{proj}_{\overrightarrow{V_2}/\overrightarrow{V_1}} = \frac{2}{\sqrt{6}} . \frac{\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}}{\sqrt{6}} = \frac{1}{3}(2\vec{\imath} + \vec{\jmath} - \vec{k})$$

$$\boxed{\overrightarrow{proj}_{\overrightarrow{V_2}/\overrightarrow{V_1}} = \frac{1}{3}(2\vec{\imath} + \vec{\jmath} - \vec{k})}$$

Exercise 4

1. The components and magnitudes of the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$

$$\overrightarrow{AB}\begin{pmatrix} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{pmatrix} \Rightarrow \overrightarrow{AB}\begin{pmatrix} -1-1 \\ 2-0 \\ 1-(-1) \end{pmatrix} \Rightarrow \overrightarrow{AB}\begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} = -2\vec{\imath} + 2\,\vec{\jmath} + 2\vec{k}$$

$$\boxed{\overrightarrow{AB} = -2\vec{\imath} + 2\,\vec{\jmath} + 2\vec{k}}$$

$$\|\overrightarrow{AB}\| = \sqrt{(-2)^2 + 2^2 + 2^2} = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$$

$$\boxed{\|\overrightarrow{AB}\| = 2\sqrt{3}}$$

$$\overrightarrow{AC}\begin{pmatrix} x_C - x_A \\ y_C - y_A \\ z_C - z_A \end{pmatrix} \Rightarrow \overrightarrow{AC}\begin{pmatrix} 2-1 \\ 1-0 \\ 3-(-1) \end{pmatrix} \Rightarrow \overrightarrow{AC}\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \vec{\imath} + \vec{\jmath} + 4\vec{k}$$

$$\boxed{\overrightarrow{AC} = \vec{\imath} + \vec{\jmath} + 4\vec{k}}$$

$$\|\overrightarrow{AC}\| = \sqrt{1^2 + 1^2 + 4^2} = \sqrt{18} = 3\sqrt{2}$$

$$\boxed{\|\overrightarrow{AC}\| = 3\sqrt{2}}$$

$$\overrightarrow{AD}\begin{pmatrix} x_D - x_A \\ y_D - y_A \\ z_D - z_A \end{pmatrix} \Rightarrow \overrightarrow{AC}\begin{pmatrix} 0-1 \\ 1-0 \\ 0-(-1) \end{pmatrix} \Rightarrow \overrightarrow{AD}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = -\vec{\imath} + \vec{\jmath} + \vec{k}$$

$$\boxed{\overrightarrow{AD} = -\vec{\imath} + \vec{\jmath} + \vec{k}}$$

$$\|\overrightarrow{AD}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\boxed{\|\overrightarrow{AD}\| = \sqrt{3}}$$

2. The projection and the vector projection of $\overrightarrow{AB}$ on $\overrightarrow{AC}$:

a) The projection of $\overrightarrow{AB}$ on $\overrightarrow{AC}$

$$proj_{\,\overrightarrow{AB}/\overrightarrow{AC}} = \frac{\overrightarrow{AB}.\overrightarrow{AC}}{\|\overrightarrow{AC}\|}$$

$$\overrightarrow{AB}.\overrightarrow{AC} = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = -2 + 2 + 8 = 8$$

$$proj_{\,\overrightarrow{AB}/\overrightarrow{AC}} = \frac{\overrightarrow{AB}.\overrightarrow{AC}}{\|\overrightarrow{AC}\|} = \frac{8}{3\sqrt{2}}$$

$$\boxed{proj_{\,\overrightarrow{AB}/\overrightarrow{AC}} = \frac{8}{3\sqrt{2}}}$$

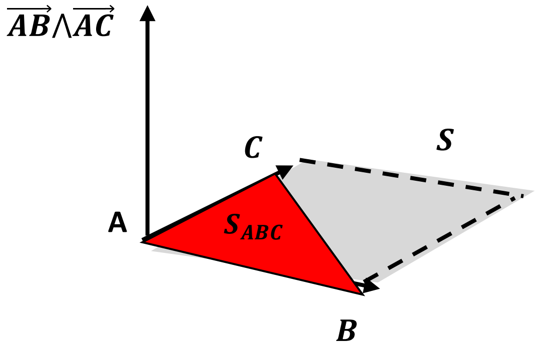
b) The vector projection of $\overrightarrow{AB}$ on $\overrightarrow{AC}$:

$$\overrightarrow{proj}_{\overrightarrow{AB}/\overrightarrow{AC}} = proj_{\overrightarrow{AB}/\overrightarrow{AC}} \,.\, \vec{u}_{\overrightarrow{AC}} = \frac{\overrightarrow{AC}.\left(\overrightarrow{AB}.\overrightarrow{AC}\right)}{\left\|\overrightarrow{AC}\right\|^2}$$

$$\Rightarrow \overrightarrow{proj}_{\overrightarrow{AB}/\overrightarrow{AC}} = 8.\frac{\left(\vec{\imath}+\vec{\jmath}+4\vec{k}\right)}{18} = \frac{4}{9}\left(\vec{\imath}+\vec{\jmath}+4\vec{k}\right)$$

$$\boxed{\overrightarrow{proj}_{\overrightarrow{AB}/\overrightarrow{AC}} = \frac{4}{9}\left(\vec{\imath}+\vec{\jmath}+4\vec{k}\right)}$$

3. The surface (area) S_{ABC} of triangle ABC:



$$S = \left\|\overrightarrow{AB}\wedge\overrightarrow{AC}\right\|$$

$$\overrightarrow{AB}\wedge\overrightarrow{AC} = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ -2 & 2 & 2 \\ 1 & 1 & 4 \end{vmatrix}$$

$$\overrightarrow{AB}\wedge\overrightarrow{AC} = (8-2)\vec{\imath} - (-8-2)\vec{\jmath} + (-2-2)\vec{k}$$

$$\overrightarrow{AB}\wedge\overrightarrow{AC} = 6.\vec{\imath} + 10.\vec{\jmath} - 4.\vec{k}$$

$$\left\|\overrightarrow{AB}\wedge\overrightarrow{AC}\right\| = \sqrt{6^2+10^2+(-4)^2} = \sqrt{152}$$

$$\boxed{S_{ABC} = \frac{S}{2} = \frac{\sqrt{152}}{2} \quad (su)}$$

- the volume constituted by $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$:

$$\overrightarrow{AD}\left(\overrightarrow{AB}\wedge\overrightarrow{AC}\right) = \begin{vmatrix} -1 & -1 & 1 \\ -2 & 2 & 2 \\ 1 & 1 & 4 \end{vmatrix} = 0$$

V=0, either the 3 vectors are in the same plane

Exercise 5

$\vec{V} = \vec{\imath} + 2\,\vec{\jmath} + 3\vec{k}$ which passes through the point A(3, 4, 2).

1. The moment of the vector $\vec{V}$ relative to the origin O

$$\vec{M}_{\vec{V}/O} = \overrightarrow{OA}\wedge\vec{V}$$

$$\overrightarrow{OA}\begin{pmatrix} x_A - x_O \\ y_A - y_O \\ z_A - z_O \end{pmatrix} \Rightarrow \overrightarrow{OA}\begin{pmatrix} 3-0 \\ 4-0 \\ 2-0 \end{pmatrix} \Rightarrow \overrightarrow{OA}\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$$

$$\vec{M}_{\vec{V}/O} = \overrightarrow{OA}\wedge\vec{V} = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ 3 & 4 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

$$\vec{M}_{\vec{V}/O} = (4.3-2.2).\vec{\imath} - (3.3-2.1).\vec{\jmath} + (3.2-4.1).\vec{k}$$

$$\boxed{\vec{M}_{\vec{V}/O} = 8.\vec{\imath} - 7\vec{\jmath} + 2.\vec{k}}$$

❋ Moment of $\vec{V}$ relative to OX:

$$\vec{M}_{\vec{V}/(OX)} = \vec{M}_{\vec{V}/O}.\vec{\imath} = \begin{pmatrix} 8 \\ 7 \\ 2 \end{pmatrix}.\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 8$$

❋ Moment of $\vec{V}$ relative to OY:

$$\vec{\mathcal{M}}_{\vec{V}/(Oy)} = \vec{\mathcal{M}}_{\vec{V}/O}.\vec{j} = \begin{pmatrix} 8 \\ 7 \\ 2 \end{pmatrix}.\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

2) Moment of vector $\vec{V}$ relative to point B (3, 6, 0)

$$\vec{\mathcal{M}}_{\vec{V}/B} = \overrightarrow{BA} \wedge \vec{V}$$

$$\overrightarrow{BA}\begin{pmatrix} x_A - x_B \\ y_A - y_B \\ z_A - z_B \end{pmatrix} \Rightarrow \overrightarrow{BA}\begin{pmatrix} 3-3 \\ 4-6 \\ 2-0 \end{pmatrix} \Rightarrow \overrightarrow{BA}\begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$$

$$\vec{\mathcal{M}}_{\vec{V}/B} = \overrightarrow{BA}\wedge\vec{V} = \begin{vmatrix} \vec{\imath} & -\vec{j} & \vec{k} \\ 0 & -2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = (-10).\vec{\imath} + 2.\vec{j} + 2.\vec{k}$$

$$\boxed{\vec{\mathcal{M}}_{\vec{V}/B} = -10.\vec{\imath} + 2.\vec{j} + 2.\vec{k}}$$

3) Moment of $\vec{V}$ relative to (Δ) :

$$\vec{\mathcal{M}}_{\vec{V}/(\Delta)} = \vec{\mathcal{M}}_{\vec{V}/B}.\vec{u} = \begin{pmatrix} -10 \\ 2 \\ 2 \end{pmatrix}.\begin{pmatrix} -1/\sqrt{2} \\ 1/2 \\ 1/2 \end{pmatrix} = \frac{10}{\sqrt{2}} + 1 + 1$$

$$\boxed{\vec{\mathcal{M}}_{\vec{V}/(\Delta)} = \frac{10}{\sqrt{2}} + 2}$$