Chapter 1. Controller design in the frequency domain

I.1 Introduction

The role of the controller is to deliver a control signal u(t) that "forces" the system to accomplish two essential tasks: first, it must stabilize the closed-loop system if it was not stable in open-loop, and then, it must bring the system's output to match the setpoint (the desired value). There are different types of controllers and various methods for designing them. Context: In the following, we will consider the classical control loop represented by the following figure:

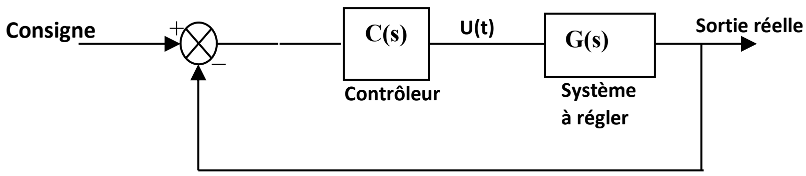


Fig 1.1 Classic control loop

I.2 Specifications in the frequency domain

The synthesis of controllers in the frequency domain requires knowing the frequency specifications of the system to be regulated in order to determine its performance. The following figure shows a typical frequency response of a closed-loop system, "the Bode plot."

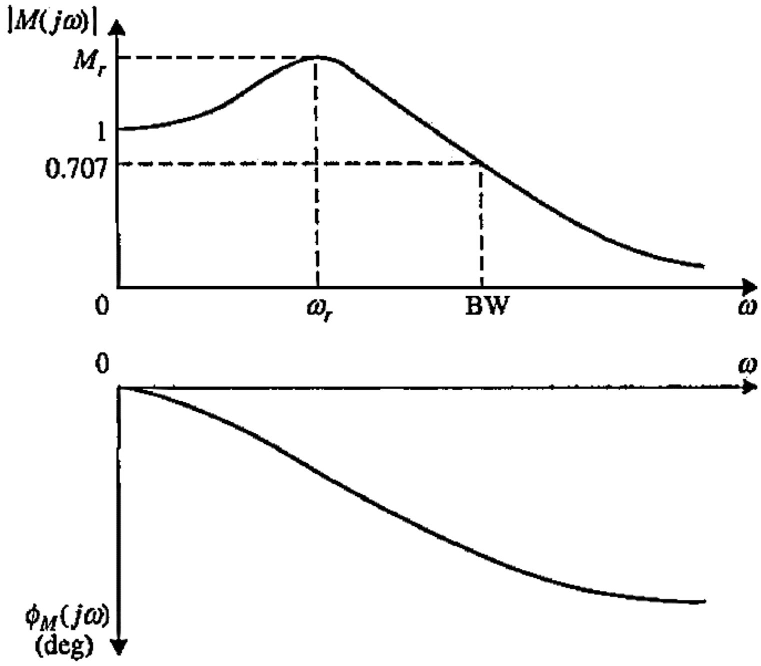


Fig 1.2 Typical Bode plot of a closed-loop system

We can distinguish the following specifications:

a- The resonance peak M_r

It is the maximum value of the modulus of the closed-loop transfer function of the servo system $|M(jw)|$, this value gives us an indication of the relative stability of the system, the larger it is, the greater the overshoot (in the step response), generally this value should be between 1.1 and 1.5.

b- The resonance frequency ω_r

It is the frequency corresponding to the resonance peak M_r.

c- The bandwidth BW

It is the frequency at which $|M(jw)| = 0.707$ (~3 db), it gives us an indication of the transient response of the system, the higher it is, the smaller the rise time of the system's step response, because the high frequencies pass. If this frequency is small, only low frequencies pass and the system's response becomes slow. The bandwidth also indicates the robustness of the system, that is, its ability to remain stable in the face of parametric variations.

I.3 Frequency specifications of a second-order system

We recall that the general form of the transfer function of a second-order system is as follows:

$$H(s) = \frac{1}{1 + \frac{2\zeta}{\omega_n} s + \frac{1}{\omega_n^2} s^2} \qquad (1.1)$$

ζ damping factor, ω_n undamped natural pulsation.

The frequency specifications of a second-order system are given by the following table:

Bande passante(Bandwidth)	$BW = \omega_n \left[(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2} \right]^{1/2}$
Pulsation de résonance	$\omega_r = \omega_n \sqrt{1 - 2\xi^2}$
Pulsation de coupure	$\omega_c = \omega_n \sqrt{1 - 2\xi^2 + \sqrt{1 + (1 - 2\xi^2)^2}}$
Facteur de résonance	$m = \frac{1}{2\xi \sqrt{1 - \xi^2}}$
Facteur de qualité	$Q = \frac{1}{2\xi}$
En décibels	$M_{dB} = 20 \log m$
	$Q_{dB} = 20 \log Q$
On a aussi	$Q = \frac{1}{2} \frac{1}{\sqrt{1 - \left(\frac{\pi}{t_{pic} \omega_n} \right)}}$

I.4 The PID controller

It is the most widely used controller in the industry, due to its simplicity and ease of adjustment. It is composed of three actions: Proportional, Integral, and Derivative. Furthermore, there are other controllers that combine more or less one of these actions, such as:

I.4.1 The proportional action controller:

This controller acts as an amplifier, its output is directly proportional to the input error, its functional diagram is as follows:

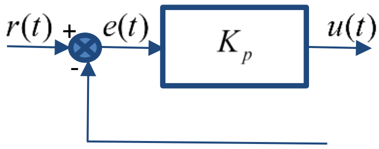


Fig 1.3 Functional diagram of the Proportional controller

His control law is given by:

$$u(t) = K_p e(t) \tag{1.2}$$

K_p proportional gain

His transfer function is given by:

$$C_p(s) = \frac{U(s)}{E(s)} = K_p \tag{1.3}$$

The Bode plot of this controller (for $K_p = 1$) is represented by the following figure:

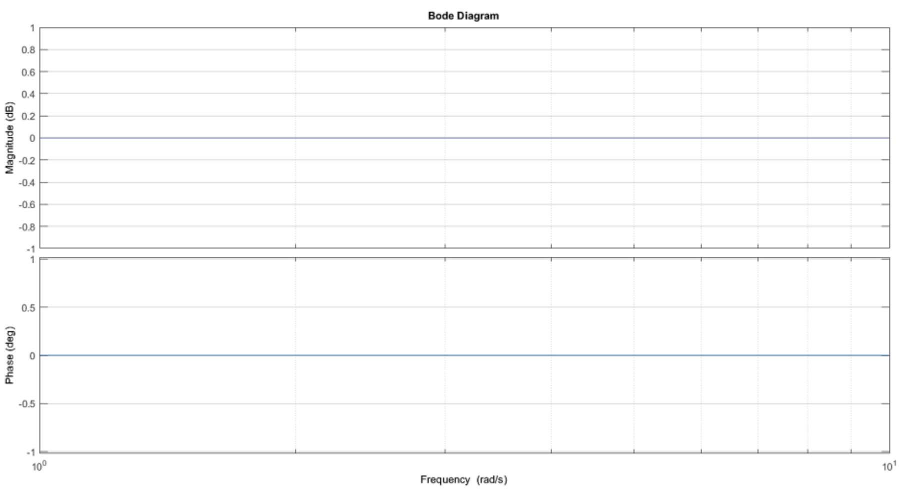


Fig 1.4 Bode plot of the Proportional controller

His task is to adjust the initial static gain of the system:

- If $K_p < 1$, it acts as an attenuator, which enhances the stability of the system and reduces overshoot in closed loop. On the other hand, this has a negative effect on speed and accuracy.
- If $K_p > 1$, it acts as an amplifier, which increases the speed and accuracy in a closed loop, however, it increases the overshoot and degrades the system's stability.

I.4.2 Proportional and Integral Action Controller

In addition to the proportional action, it adds a zero pole (integrator) to the control loop, its functional diagram is as follows:

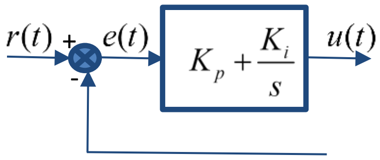


Fig 1.5 Functional diagram of the proportional-integral controller

His control law is given by:

$$u(t) = K_p + K_i \int_0^t e(t)dt \qquad (1.4)$$

K_i gain of the integral action.

His transfer function is as follows:

$$C_{pi}(s) = \frac{U(s)}{E(s)} = K_p + \frac{K_i}{s} \qquad (1.5)$$

The Bode plot of this controller (for $K_p = K_i = 1$) is represented by the following figure:

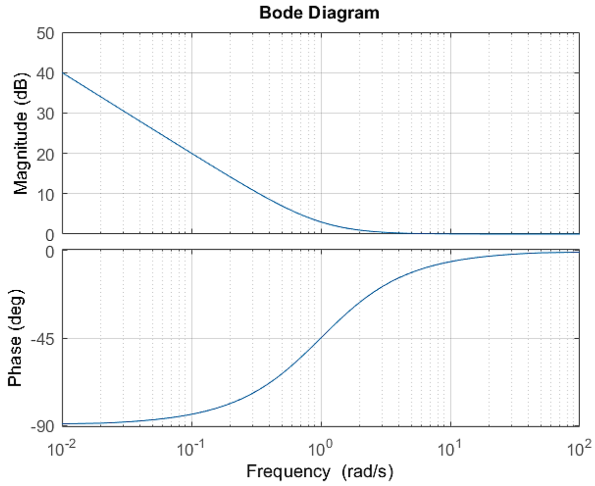


Fig 1.6 Bode plot of the Proportional Integral controller

The main frequency characteristics of a PI controller:

- **Low-frequency gain** : At low frequencies, the gain of the PI controller is high. This means that the PI controller amplifies low-frequency signals, making it effective for correcting long-term regulation errors, such as steady-state regulation errors.

- **Incline of -20 dB/decade in amplitude** : The PI controller has a slope of -20 dB/decade in amplitude in the low-frequency region. This means that the amplitude of the response decreases by 20 dB for each decade of frequency increase. This slope indicates that the PI controller provides an integral correction, which is beneficial for eliminating steady-state regulation errors. There is also a cutoff frequency in integration that depends on K_p and K_i, beyond which the integral gain begins to decrease.

- **Phase shift of -90 degrees**: In the low-frequency region, the PI controller introduces a phase shift of -90 degrees (lagging phase) between the input and the output. This indicates that the PI controller reacts reactively to long-term regulation errors.

  Stability : The PI controller is generally stable, which means it does not significantly amplify high frequencies. However, it is important to properly adjust the parameters K_p and K_i to avoid unstable or oscillating responses.

The main effects of this controller on the control loop are as follows:

- ✓ Elimination of static error
- ✓ Increase in the overall gain of the system at low frequencies.
- ✓ Increase in the system's speed

<u>Homework:</u>
Show that the addition of a pure integrator in a closed control loop has the effect of eliminating static error.

I.4.3 Proportional and Derivative Action Controller

He combines a proportional action with a derivative action, his functional diagram is as follows:

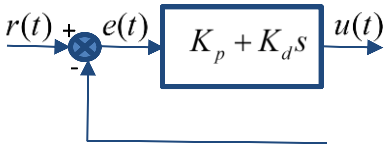


Fig 1.7 Functional diagram of the proportional derivative controller

K_d derived gain

His control law is given by:

$$u(t) = K_p + K_p \frac{de(t)}{dt} \tag{1.6}$$

His transfer function is given by:

$$C_{PD}(s) = \frac{U(p)}{E(p)} = K_p + K_d s \tag{1.7}$$

As can be noticed, this transfer function is non-causal (degree of the numerator > degree of the denominator), hence the use of a filter

The Bode plot of this controller (for K_p= 2 and K_d= 1) is given by:

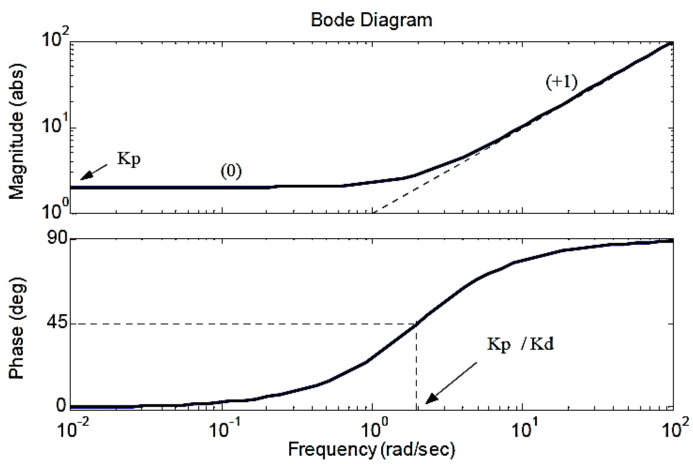


Fig 1.8 Bode plot of the Proportional Derivative

In order to understand the main effect of the derivative action, observe the following figure:

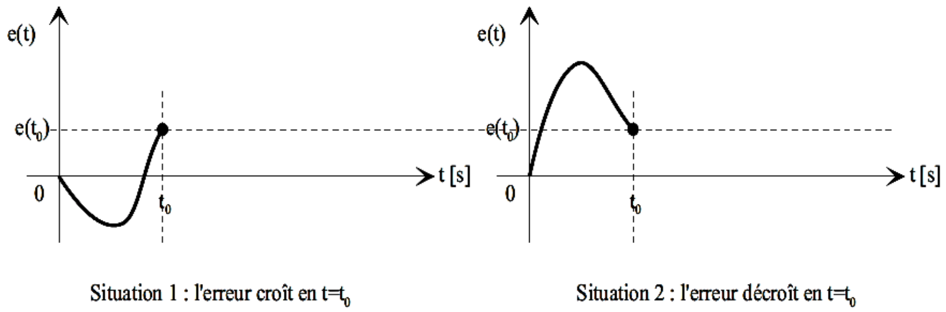


Fig 1.9 Two identical situations of a temporal response of a system

At time $t=t_0$, the amplitude of the error in both situations is the same, but in the first situation, the response increases, while in the second situation, it decreases. This raises the following issue: how can the controller know whether the response is increasing or decreasing so that it can take the necessary measures?
The derivative of the response e(t) with respect to time at instant t0 allows us to know if the response is increasing or decreasing, hence the usefulness of the derivative action. The latter has other effects, namely:

- ✓ Increase in damping and reduction of overshoot.
- ✓ Increase in the speed of the temporal response (reduction of the rise time).
- ✓ Increase in bandwidth.
- ✓ Improvement of phase margins and gain margin.

Its main negative effect can be observed at its Bode location:

- ✓ Accentuation of high-frequency noises.

I.4.4 Proportional-Integral-Derivative Controller:

His functional diagram is as follows:

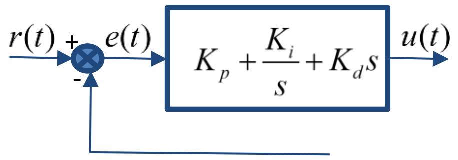


Fig 1.10 Functional diagram of the proportional integral derivative controller

He combines the three actions: proportional, integral, and derivative, his control law is given by:

$$u(t) = K_p e(t) + K_i \int_0^t e(t)dt + K_d \frac{de(t)}{dt} \qquad (1.8)$$

His transfer function is given by:

$$C_{PID}(s) = \frac{U(s)}{E(s)} = K_p + \frac{K_i}{s} + K_d s \quad (1.9)$$

The PID controller benefits from the advantages of each action that composes it (speed for proportional and derivative actions, elimination of static error for integral action, etc.). There are several methods for adjusting the three parameters of the PID controller, notably time-response-based methods such as the Ziegler-Nichols method…etc. In what follows, we will study the frequency adjustment methods.

I.4.5 Phase-lag compensator(Correcteur à retard de phase)

The phase-lag compensator primarily aims to increase the gain at low frequencies. It is therefore used to increase the accuracy of a servo system. Context: Its transfer function is given by:

$$C(s) = \frac{a(1+T.s)}{(1+aT.s)} \quad (1.10)$$

with $a > 1$.

We notice that he has two cutoff pulses $\frac{1}{T} > \frac{1}{aT}$.

Indeed, by setting the parameter T to a sufficiently low value, the correction action as well as the negative phase shift (phase lag) introduced by the compensator will only be effective at low frequencies, so there will be no influence on the stability margin, as the cutoff frequencies at 0 dB (ω_{co}) are generally located at higher frequencies.
To adjust the phase-lag compensator, we will choose the value of that will achieve the desired resulting static gain, and then we will choose such that T tel que $\frac{1}{T} \ll \omega_{co}$.

I.4.6 Phase-lead compensator(Correcteur à avance de phase)

The phase-lead compensator primarily serves to increase the phase margin of a system. Its transfer function is given by:

$$C(s) = \frac{1+aT.s}{1+T.s} \quad (1.11)$$

with $a > 1$.

We notice that he has two cutoff pulses $\frac{1}{T} > \frac{1}{aT}$.

To the pulse $\omega_{max} = \frac{1}{T\sqrt{a}}$, the phase shift is maximal $\varphi_{max} = \arcsin\frac{a-1}{a+1}$. The corrective action will then consist of making ω_{max} with the cutoff pulse at 0 dB ω_{co} of the system to correct and adjust φ_{max} in order to obtain the desired phase margin (phase increase).

I.5 Frequency adjustment methods

Frequency methods are based on frequency responses; they involve adjusting the parameters of the PID controller while adhering to a specification document that includes frequency criteria such as gain margins, phase margins, bandwidth, and other similar specifications.

I.5.1 Bode Tuning Method

It is based on the analysis of the frequency responses of the control system and allows for precise adjustment of the PID parameters to achieve the desired performance. It consists of first eliminating the largest time constant of the open-loop system (dominant pole), and then adjusting the proportional gain K_p according to the desired gain margins A_m and phase margins φ_m.

Let the general form of a system's transfer function be:

$$G(s) = \frac{K_a}{s^\alpha} \cdot \frac{(1 + T_1 . s).(1 + T_2 . s) + \cdots \ldots \ldots}{(1 + \tau_1 . s).(1 + \tau_2 . s) + \cdots \ldots \ldots}$$

a) For a PI controller

Let's take the transfer function of a controller PI : $C_{pi}(s) = K_p . \frac{(1+T_i . s)}{T_i . s}$(this is another form of the PI controller), the open-loop transfer function of the system with the controller will therefore be:

$$G_{Bo}(s) = C_i(s). G(s) = K_p . \frac{(1 + T_i . s)}{T_i . s} . \frac{K_a .(1 + T_1 . s).(1 + T_2 . s) + \cdots}{(1 + \tau_1 . s).(1 + \tau_2 . s) + \cdots}$$

with : $K_o = \frac{K_p . K_a}{T_i}$.

To eliminate the largest time constant of the system τ_{max} (dominant pole), we put $T_i = \tau_{max}$. The open loop transfer function will then become:

$$G_{Bo}(s) = \frac{(1 + \tau_{max} . s)}{s} . \frac{K_o .(1 + T_1 . s).(1 + T_2 . s) + \cdots}{(1 + \tau_1 . s).(1 + \tau_2 . s) + \cdots}$$

b) For a PD controller

Let be the transfer function of a PD controller: $C_{pd}(s) = K_p .(1 + T_d . s)$ (this is another form of the PD controller), the open loop transfer function of the system with the controller will therefore be:

$$G_{Bo}(s) = C_d(s). G(s) = K_p .(1 + T_d . s). \frac{K_a .(1 + T_1 . s).(1 + T_2 . s) + \cdots}{(1 + \tau_1 . s).(1 + \tau_2 . s) + \cdots}$$

with : $K_o = K_p . K_a$

To eliminate the largest time constant from the system τ_{max} (dominant pole), we put $T_d = \tau_{max}$. The open-loop transfer function will then become:

$$G_{Bo}(s) = (1 + \tau_{max1}.s).\frac{K_o.(1 + T_1.s).(1 + T_2.s) + \cdots}{(1 + \tau_1.s).(1 + \tau_2.s) + \cdots}$$

c) **For a PID controller**

d) Let the transfer function of a PID controller be: $C_{pid}(s) = K_p.\frac{(1+T_i.s+T_i.T_d.s^2)}{T_i.s}$. This is another form of the PID controller, so the open-loop transfer function of the system will be:

$$G_{Bo}(s) = C_{pid}(s).G(s) = K_p.\frac{(1 + T_i.s + T_i.T_d.s^2)}{T_i.s}.\frac{K_a.(1 + T_1.s).(1 + T_2.s) + \cdots}{(1 + \tau_1.s).(1 + \tau_2.s) + \cdots}$$

with : $K_o = \frac{K_p.K_a}{T_i}$

The method consists of first eliminating the two largest time constants of the system τ_{max1} and τ_{max2} (dominant poles) by setting:

$$(1 + T_i.s + T_i.T_d.s^2) = (1 + \tau_{max1}.s).(1 + \tau_{max2}.s)$$

The open-loop transfer function will then become:

$$G_{Bo}(s) = \frac{(1 + \tau_{max1}.s).(1 + \tau_{max2}.s)}{s}.\frac{K_o.(1 + T_1.s).(1 + T_2.s) + \cdots}{(1 + \tau_1.s).(1 + \tau_2.s) + \cdots}$$

<u>Second step: Gain adjustment K_p</u>

Let $G_{bo}(s)$ be a transfer function of a system corrected by an open-loop controller:

$$G_{bo}(s) = C(s).G(s) = \frac{K_o}{s^\alpha}.\frac{(1 + T_1.s).(1 + T_2.s) + \cdots}{(1 + \tau_1.s).(1 + \tau_2.s) + \cdots}$$

It is then about:

1) Plot the Bode diagram of $G(s)$ for $K_o = 1$.

2) To determine the pulsation w_{co}, that is: arg{G(jw_{co})} = −180° + φ_m, where φ_m is the desired phase margin (generally 45°).
3) Record the loop gain |G(jw_{co})| at this pulse.
4) Calculate the gain K_o to apply to G(p) so that: K_o. |G(jw_{co})|=1 db, that is: $K_o = \frac{1}{|G(jw_{co})|}$.
5) Deduce the value of K_p

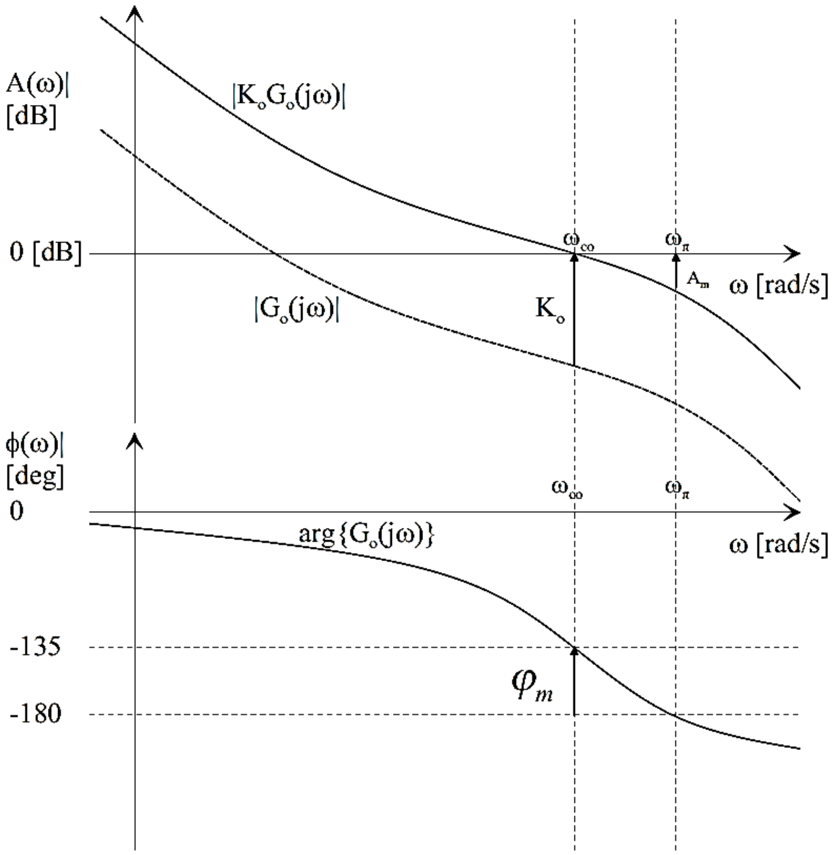


Fig 1.11 Illustration of the Bode method

Exemple :

Sois la fonction de transfert d'un système :

$$G(s) = \frac{100}{(1+0.01.s).(1+0.001.s).(1+0.0001.s)}$$

On veut commander ce systéme par un contrôleur PID de fonction de transfert :

$$C_{pid}(s) = K_p.\frac{(1+T_i.s+T_i.T_d.s^2)}{T_i.s}$$

1- **T_i et T_d sont trouvés en résolvant l'équation :**

$$(1+T_i.s+T_i.T_d.s^2) = (1+\tau_{max1}.s).(1+\tau_{max2}.s) =$$

$$(1+0.01.s).(1+0.001.s) = (1+0.011.s+0.00001.s^2)$$

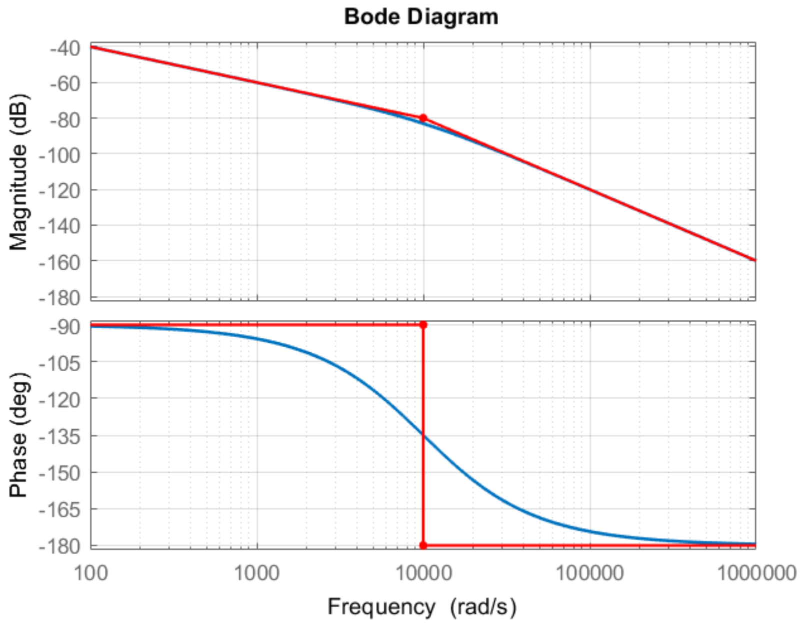
D'où : $T_i = 0.011$ et $T_i.T_d = 0.00001 \rightarrow T_d = 0.00091.$

2- **Réglage de K_p**

La fonction de transfert en boucle ouverte est donnée par :

$$G_{bo}(s) = C_{pid}(s).G(s) = \frac{K_o}{s}.\frac{1}{(1+0.0001.s)}, \text{ avec : } K_o = \frac{K_p.K_a}{T_i} = \frac{K_p.100}{0.011}$$

Le diagramme asymptotique de Bode de $G_{bo}(s)$ (pour $K_o = 1$) est représenté par la figure suivante:



On remarque que si on veut avoir une marge de phase de 45° , alors :

$$\arg\{G(jw_{co})\} = -180° + 45° = -135°$$

D'après le tracé w_{co} = 0.0001 rad/sec. Et le gain pour cette pulsation est égale à -80 db, donc :

$K_o = \frac{1}{|G(jw_{co})|}$. On a alors : 20.log($|G(jw_{co})|$)= 80, donc $|G(jw_{co})| = 10^4$.

Enfin on a $K_o = \frac{K_p.100}{0.011}$, donc : $K_p = \frac{K_o.0.011}{100} = \frac{10^4.0.011}{100} = 1.1$