Modelling and Identification of Systems

Semester 5 — UEF 3.1.1

Credits: 2 Coefficient: 1

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Contents

1. Modelling

1.1 Learning Objectives

At the end of this chapter, the student should be able to:

- Understand the concept and role of a model in engineering.
- Distinguish between physical (knowledge-based) and empirical models.
- Derive mathematical models of basic physical systems (mechanical, electrical, fluidic, thermal).
- Represent systems using differential equations, transfer functions, and block diagrams.

1.2 Introduction to Modelling

1.2.1 Definition of a Model

A **model** is a simplified representation of a physical system that captures its essential behavior while ignoring insignificant details. Mathematically, a model relates the *inputs* and *outputs* of a system:

$$y(t) = f(u(t), \dot{u}(t), \ldots, y(t), \dot{y}(t), \theta) \tag{1.1}$$

where:

- $u(t)$: input (excitation, control signal)
- $y(t)$: output (measured response)
- θ: parameters describing system properties

A model should be:

- **Representative** — captures real dynamics

- **Simple enough** — for analysis and control
- **Adapted** — to available measurements

1.2.2 Types of Models

1. **Physical or Knowledge-Based Models**: Based on laws of physics (Newton, Kirchhoff, etc.)
2. **Empirical or Data-Based Models**: Derived from experimental data without explicit physical laws
3. **Hybrid Models**: Combine physical structure with parameter estimation from data

1.3 System Representation

A dynamic system can be represented in several ways:

Representation	**Description**	**Example**
Differential Equation	Relates derivatives of input and output	$a_1\dot{y} + a_0 y = b_1\dot{u} + b_0 u$
Transfer Function	Ratio of Laplace-transformed output to input	$G(s) = \frac{Y(s)}{U(s)} = \frac{b_1 s + b_0}{a_1 s + a_0}$
State-Space Model	Describes internal dynamics	$\dot{x} = Ax + Bu, \quad y = Cx + Du$
Block Diagram	Graphical interconnection of subsystems	Useful for control design

1.4 Knowledge-Based Modelling

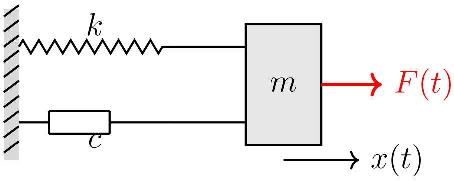
This approach uses **physical laws** such as:

- Newton's second law for mechanical systems
- Kirchhoff's laws for electrical systems
- Continuity and energy balance equations for fluidic and thermal systems

1.4.1 Mechanical Systems

Example: Mass–Spring–Damper System

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t) \tag{1.2}$$



Transfer function:

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} \tag{1.3}$$

State-Space Representation

To express the same system in state-space form, we define suitable *state variables* that describe the system's dynamics.

1. Choice of states

Let:

$$x_1(t) = x(t) \quad \text{(displacement)}, \qquad x_2(t) = \dot{x}(t) \quad \text{(velocity)}.$$

Then,

$$\dot{x}_1(t) = x_2(t).$$

From the equation of motion

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t),$$

we express the acceleration:

$$\ddot{x}(t) = -\frac{c}{m}\dot{x}(t) - \frac{k}{m}x(t) + \frac{1}{m}F(t).$$

Substitute $x_2 = \dot{x}$:

$$\dot{x}_2(t) = -\frac{c}{m}x_2(t) - \frac{k}{m}x_1(t) + \frac{1}{m}F(t).$$

2. Matrix form

The state equations can now be written in vector–matrix form as:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} F(t).$$

The output equation (displacement as output) is:

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} F(t).$$

3. Compact form

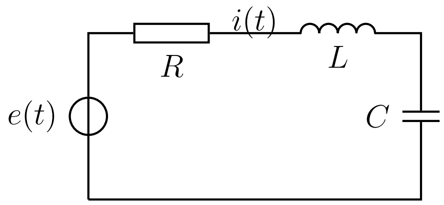
$$\boxed{\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t), \\ y(t) &= C\,x(t) + D\,u(t), \end{aligned}}$$

where $x(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $u(t) = F(t)$, and the matrices are:

$$A = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = [\,0\,].$$

1.4.2 Electrical Systems

Example: Series RLC Circuit



Transfer function.

From Kirchhoff's voltage law (KVL) around the loop:

$$e(t) = v_R(t) + v_L(t) + v_C(t) = R\,i(t) + L\frac{di(t)}{dt} + v_C(t).$$

Using $v_C(t) = \dfrac{1}{C}\int i(t)\,dt$ (or in Laplace $V_C(s) = \dfrac{1}{Cs}I(s)$), take the Laplace transform (zero initial conditions):

$$E(s) = \left(R + Ls + \frac{1}{Cs}\right) I(s).$$

Hence the (input-to-current) transfer function is

$$\boxed{G(s) = \frac{I(s)}{E(s)} = \frac{1}{R + Ls + \dfrac{1}{Cs}}}.$$

State-space Representation

Choose states:

$$x_1(t) = i(t) \quad \text{(inductor current)}, \qquad x_2(t) = v_C(t) \quad \text{(capacitor voltage)}.$$

Use two relations:

- Inductor voltage: $v_L(t) = L\dot{i}(t)$.
- Capacitor current: $i(t) = C\dot{v}_C(t)$, so $\dot{v}_C(t) = \frac{1}{C}i(t)$.

From KVL: $e(t) = Ri(t) + L\dot{i}(t) + v_C(t)$. Solve for $\dot{i}(t)$:

$$L\dot{i}(t) = e(t) - Ri(t) - v_C(t) \quad \Rightarrow \quad \dot{x}_1(t) = -\frac{R}{L}x_1(t) - \frac{1}{L}x_2(t) + \frac{1}{L}e(t).$$

And from the capacitor relation:

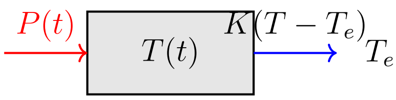
$$\dot{x}_2(t) = \frac{1}{C}x_1(t).$$

Write in matrix form with input $u(t) = e(t)$ and output $y(t)$ (choose output as current i or capacitor voltage; here we take $y = i$):

$$\dot{x}(t) = \underbrace{\begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}}_{B} u(t), \qquad y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x(t) + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} u(t).$$

1.4.3 Thermal Systems

Example: Heating a Metal Block



Variables and parameters:

$$\begin{aligned} T(t) &: \text{ temperature of the object (°C)} \\ T_e &: \text{ ambient temperature (°C), assumed constant} \\ P(t) &: \text{ input heat power (W)} \\ C &: \text{ thermal capacity (J/°C)} \\ K &: \text{ overall heat transfer coefficient (W/°C)} \end{aligned}$$

Energy balance:

The rate of energy stored in the object equals the heat input minus the heat lost to the environment:

$$C\frac{dT(t)}{dt} = P(t) - K\big(T(t) - T_e\big)$$

Derivation of the transfer function:

1. Take the Laplace transform assuming zero initial conditions:

$$CsT(s) = P(s) - K[T(s) - T_e/s]$$

2. Since T_e is constant, its Laplace transform term T_e/s represents a steady offset; for simplicity, consider the deviation variable $\theta(t) = T(t) - T_e$, which gives:

$$C\frac{d\theta(t)}{dt} + K\theta(t) = P(t)$$

3. Taking the Laplace transform again:

$$(Cs + K)\Theta(s) = P(s)$$

Thus, the transfer function between input power $P(s)$ and temperature variation $\Theta(s)$ is:

$$G(s) = \frac{\Theta(s)}{P(s)} = \frac{1}{Cs + K}$$

State-space form: Let $x_1(t) = \theta(t) = T(t) - T_e$. Then,

$$\dot{x}_1(t) = -\frac{K}{C}x_1(t) + \frac{1}{C}P(t)$$

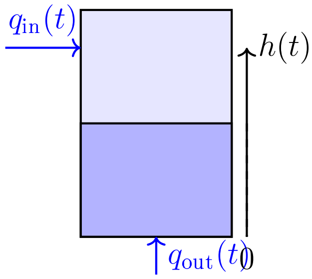
$$y(t) = x_1(t)$$

Hence, the state-space model is:

$$\boxed{\begin{aligned} \dot{x}_1 &= -\frac{K}{C}x_1 + \frac{1}{C}P(t) \\ y &= x_1 \end{aligned}}$$

1.4.4 Fluidic Systems

Example: Tank with Constant Cross-Section



Variables and parameters:

$$\begin{aligned} A &: \text{ cross-sectional area of the tank } (\text{m}^2) \\ h(t) &: \text{ liquid height in the tank (m)} \\ q_{\text{in}}(t) &: \text{ inflow rate } (\text{m}^3/\text{s}) \\ q_{\text{out}}(t) &: \text{ outflow rate } (\text{m}^3/\text{s}) \\ K &: \text{ outflow constant } (\text{m}^{2.5}/\text{s}) \end{aligned}$$

Dynamic model:

$$A\frac{dh(t)}{dt} = q_{\text{in}}(t) - q_{\text{out}}(t), \quad \text{with} \quad q_{\text{out}}(t) = K\sqrt{h(t)}$$

Linearization:

At steady state $h(t) = h_0$ and $q_{\text{in}}(t) = q_{\text{out}}(t) = K\sqrt{h_0}$. Define small variations around equilibrium:

$$\Delta h(t) = h(t) - h_0, \quad \Delta q_{\text{in}}(t) = q_{\text{in}}(t) - q_{\text{in},0}$$

Linearizing $q_{\text{out}}(t)$ around h_0 gives:

$$\Delta q_{\text{out}}(t) \approx \frac{K}{2\sqrt{h_0}}\,\Delta h(t) = K'\,\Delta h(t)$$

where $K' = \dfrac{K}{2\sqrt{h_0}}$ is the linearized outflow coefficient.

Linearized differential equation:

$$A\frac{d(\Delta h)}{dt} = \Delta q_{\text{in}}(t) - K'\,\Delta h(t)$$

Transfer function derivation:

Taking the Laplace transform:

$$As\,\Delta H(s) = \Delta Q_{\text{in}}(s) - K'\,\Delta H(s)$$

$$\Rightarrow (As + K')\Delta H(s) = \Delta Q_{\text{in}}(s)$$

$$\Rightarrow G(s) = \frac{\Delta H(s)}{\Delta Q_{\text{in}}(s)} = \frac{1/A}{s + K'/A}$$

State-space representation:
Let $x_1(t) = \Delta h(t)$ (liquid level deviation). Then:

$$\dot{x}_1(t) = -\frac{K'}{A}x_1(t) + \frac{1}{A}\Delta q_{\text{in}}(t)$$
$$y(t) = x_1(t)$$

$$\boxed{\begin{aligned} \dot{x}_1 &= -\frac{K'}{A}x_1 + \frac{1}{A}\Delta q_{\text{in}}(t) \\ y &= x_1 \end{aligned}}$$

1.5 Linearization of Nonlinear Models

Many physical systems are inherently nonlinear due to phenomena such as saturation, friction, backlash, or nonlinear restoring forces. However, nonlinear models are often difficult to analyze and control directly. To simplify analysis and design, it is common to approximate the nonlinear model by a linear one around a given operating point.

1.5.1 Concept of Linearization

Consider a nonlinear dynamical system described by

$$\dot{x} = f(x, u) \tag{1.4}$$

where $x \in \mathbb{R}^n$ is the state vector and $u \in \mathbb{R}^m$ is the control input. Let (x_0, u_0) be an equilibrium point of the system such that $f(x_0, u_0) = 0$.

To analyze the behavior of the system in the neighborhood of this point, small perturbations are introduced:

$$\Delta x = x - x_0, \quad \Delta u = u - u_0$$

Expanding $f(x, u)$ in a first-order Taylor series around (x_0, u_0) yields:

$$f(x, u) \approx f(x_0, u_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0, u_0}(x - x_0) + \left.\frac{\partial f}{\partial u}\right|_{x_0, u_0}(u - u_0)$$

Since $f(x_0, u_0) = 0$, the linearized model becomes:

$$\Delta\dot{x} = A\,\Delta x + B\,\Delta u \tag{1.5}$$

with the Jacobian matrices:

$$A = \left.\frac{\partial f}{\partial x}\right|_{x_0,u_0}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{x_0,u_0} \tag{1.6}$$

1.5.2 Interpretation of the Linearized Model

The matrices A and B represent the local dynamics and control influence near the equilibrium point:

- A captures how small deviations in the state affect the rate of change of the system states.
- B indicates how small variations in the control input influence the state dynamics.

The resulting linear system $\Delta\dot{x} = A\Delta x + B\Delta u$ can be analyzed using classical linear techniques such as eigenvalue analysis, stability margins, frequency response, and controllability tests.

1.5.3 Validity of Linearization

The linear approximation is only valid in a sufficiently small neighborhood of (x_0, u_0). For large deviations, higher-order nonlinear effects become significant, and the linearized model no longer accurately represents the true system. Therefore, it is crucial to:

- Verify that perturbations remain small.
- Compare the linearized model response with the original nonlinear one.
- Use the linearized model primarily for local control design or stability analysis.

1.5.4 Example: Linearization of a Nonlinear Pendulum

To illustrate the linearization process, consider the nonlinear dynamics of a simple pendulum of length l and mass m:

$$\ddot{\theta} + \frac{g}{l}\sin(\theta) = 0 \tag{1.7}$$

where θ is the angular displacement and g is the gravitational acceleration.

State-Space Representation

Let

$$x_1 = \theta, \quad x_2 = \dot{\theta}$$

Then, the system can be written as:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{g}{l}\sin(x_1) \end{cases}$$

or compactly,

$$\dot{x} = \begin{bmatrix} x_2 \\ -\dfrac{g}{l}\sin(x_1) \end{bmatrix} = f(x)$$

Equilibrium Point

The equilibrium corresponds to the pendulum at rest in the vertical position:

$$x_1 = 0, \quad x_2 = 0$$

That is, $(x_0, u_0) = (0, 0)$.

Jacobian Matrices

We compute the Jacobian matrix $A = \frac{\partial f}{\partial x}$:

$$A = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{g}{l}\cos(x_1) & 0 \end{bmatrix}$$

Evaluating at the equilibrium $(x_1 = 0)$ gives:

$$A = \begin{bmatrix} 0 & 1 \\ -\dfrac{g}{l} & 0 \end{bmatrix}$$

There is no control input u, so B is omitted in this case.

Linearized Model

The linearized system around the equilibrium is:

$$\begin{cases} \Delta\dot{x}_1 = \Delta x_2 \\ \Delta\dot{x}_2 = -\dfrac{g}{l}\,\Delta x_1 \end{cases}$$

which can be written as:

$$\Delta\ddot{\theta} + \frac{g}{l}\Delta\theta = 0$$

This is a linear, second-order differential equation representing **small oscillations** around the vertical equilibrium.

Transfer Function Form

Taking the Laplace transform, the transfer function between the angular displacement and an applied torque $u(t)$ (if one were added) would be:

$$\frac{\Theta(s)}{U(s)} = \frac{1}{ls^2 + g}$$

When $u(t) = 0$, the natural frequency of the linearized pendulum is:

$$\omega_n = \sqrt{\frac{g}{l}}$$

Interpretation

The linearized model is an accurate approximation for small angles θ (typically $|\theta| < 10°$). For larger angles, the $\sin(\theta)$ term deviates significantly from θ, and nonlinear behavior such as amplitude-dependent frequency appears.

1.6 Model Validation

Once a mathematical model has been obtained, it must be validated to ensure that it accurately represents the real system. Model validation consists of comparing the model's predicted outputs with actual experimental or measured data under the same input conditions. A validated model should not only fit the observed data but also remain physically meaningful and generalize well to other operating conditions.

1.6.1 Purpose of Validation

Model validation verifies whether the identified model is suitable for analysis, simulation, and control design. It answers key questions such as:

- Does the model reproduce the system's behavior with acceptable accuracy?
- Are the estimated parameters physically coherent?
- Can the model predict the response to new inputs not used during identification?

1.6.2 Validation Methods

Several complementary techniques are used to assess model quality:

1. **Visual comparison:** Compare simulated and measured outputs in the time domain (or frequency domain). A good model shows close overlap of curves and correct dynamic behavior (overshoot, settling time, etc.).

2. **Error analysis:** Compute quantitative indicators of model accuracy, such as:

$$\text{MSE} = \frac{1}{N}\sum_{k=1}^{N}(y_{\text{meas}}(k) - y_{\text{model}}(k))^2$$

   Other useful criteria include the Mean Absolute Error (MAE) and the coefficient of determination R^2.

3. **Residual analysis:** Examine the residual signal:

$$e(k) = y_{\text{meas}}(k) - y_{\text{model}}(k)$$

   The residuals should behave like white noise — uncorrelated with past inputs and outputs. Significant correlation indicates missing dynamics or nonlinear effects.

4. **Physical consistency:** The estimated parameters (mass, resistance, capacitance, gain, etc.) should have realistic magnitudes and signs consistent with physical laws.

1.6.3 Validation Example

Suppose a first-order model

$$G(s) = \frac{K}{\tau s + 1}$$

is identified from step-response data. To validate it:

- Simulate the model response to the same step input and compare it visually with the experimental output.
- Compute the MSE between both signals.
- Check that $\tau > 0$ and K has the expected sign.

If the results are satisfactory, the model can be used for controller design. Otherwise, a more complex structure or refined identification may be required.

1.7 Conclusion

In this chapter, we introduced the fundamental principles of system modeling and representation. We examined how physical systems from various domains—mechanical, electrical, thermal, and fluidic—can be expressed through differential equations, transfer functions, and state-space models.

Emphasis was placed on understanding the relationships between system inputs, outputs, and internal states, as well as on the assumptions that allow simplification and linearization of real-world nonlinear dynamics. The importance of model validation was also highlighted, showing that a reliable model must not only fit the data but also remain physically meaningful.

The modeling techniques discussed here form the foundation for the analysis and control of dynamic systems. In the next chapter, these models will serve as the basis for studying system behavior in the time and frequency domains, and for developing appropriate control strategies.

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