CHAPTER III ANALYSIS OF SYSTEMS IN STATE SPACE

III.1 Introduction

The state representation of systems requires specific analysis methods regarding stability, controllability, and observability. In this chapter, we will address the resolution of state equations and methods for calculating the transition matrix, modal analysis (diagonalization), as well as the study of stability. Finally, we will introduce two fundamental concepts of state representation: controllability and observability.

III.2 Resolution of the equation of state

We have seen that the state equation describes the temporal dynamics of a system thru its state variables. The resolution of this equation thus allows us to determine the dynamic behavior of the system over time. For this, let's start first with the homogeneous equation $\frac{dX}{dt} = AX$; its solution is given by:

$$X(t) = e^{A(t-t_0)} X(t_0) \tag{3.1}$$

t_0 is the initial moment.

We have:

$$\frac{dx}{dt} = Ax(t) + Bu(t) \rightarrow e^{-At}\frac{dx}{dt} = e^{-At}Ax(t) + e^{-At}Bu(t) = Ae^{-At}x(t) + e^{-At}Bu(t) \rightarrow$$

$$e^{-At}\frac{dx}{dt} - Ae^{-At}x(t) = e^{-At}Bu(t) \rightarrow \frac{d}{dt}(e^{-At}x(t)) = e^{-At}Bu(t)e^{-At}$$

$$e^{-At}x(t) = e^{-At_0}x(t_0) + \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$$

And finally:

$$x(t) = \underbrace{\underbrace{e^{A(t-t_0)}x(t_0)}_{\textit{Réponse libre}} + \underbrace{\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau}_{\textit{Réponse forcée}}}_{\textit{Réponse complète}} \tag{3.2}$$

The solution $x(t)$ is expressed as the sum of the free regime solution (when no input signal is applied $u(t) = 0$), and the forced regime solution (when the initial conditions are zero).
e^{At} is called the transition matrix.

III.3 Calculation of the transition matrix

The Taylor series expansion of the matrix exponential e^{At} is given by:

$$e^{At} = I + At + \frac{A^2}{2!}t^2 + \ldots + \frac{A^n}{n!}t^n + \ldots \tag{3.3}$$

.
Or:

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n}{n!}t^n \tag{3.4}$$

This series is therefore an infinite sum that allows us to approximate the exponential function of a matrix A in terms of its powers. There are several approaches to calculating the transition matrix e^{At}, among which:

III.3.1 The Cayley-Hamilton theorem

The Cayley-Hamilton theorem offers an interesting approach to calculate e^{At} using a finite number of operations. According to this theorem, any square matrix (A) is a solution to its characteristic equation. This property is therefore used to express high powers (A) of in terms of lower powers.

If $P_A(s)$ is the characteristic polynomial of A_{nxn}, defined as:

$$P_A(s) = \det(sI - A) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{3.5}$$

So according to the Cayley-Hamilton theorem, A verifies:

$$A^n + a_{n-1}A^{n-1} + \ldots + a_1 A + a_0 I = 0 \tag{3.6}$$

From where:

$$A^n = -a_{n-1}A^{n-1} - \ldots - a_1 A - a_0 I \tag{3.7}$$

By substituting (3.7) into (3.4), we can write:

$$e^{At}=\sum_{n=0}^{\infty}\frac{A^n}{n!}t^n=\alpha_{n-1}(t)A^{n-1}+\ldots+\alpha_1(t)A+\alpha_0 I \tag{3.8}$$

Where $\alpha_0,\alpha_1,\ldots,\alpha_{n-1}$ coefficients to be determined.

When the eigenvalues of the matrix A are distinct, the calculation of the coefficients $(\alpha_0,\alpha_1,\ldots,\alpha_{n-1})$ can be simplified by using a basis of eigenvectors $(x_0,x_1,\ldots,x_{n-1})$ of A. Let's denote the corresponding eigenvalues as $(\lambda_0,\lambda_1,\ldots,\lambda_{n-1})$ respectively, and rewrite equation (3.8) applied to each eigenvector of A :

$$e^{At}x_k=\alpha_{n-1}(t)A^{n-1}x_k+\ldots+\alpha_1(t)Ax_k+\alpha_0(t)Ix_k \tag{3.9}$$

Given that $Ax_k=\lambda_k x_k$ (see chapter 2), we can write:

$$e^{At}x_k=\alpha_{n-1}(t)A^{n-1}x_k+\ldots+\alpha_1(t)Ax_k+\alpha_0(t)Ix_k=\sum_{0}^{+\infty}\frac{\lambda_k^n}{n!}t^n x_k=e^{\lambda_k t}x_k \tag{3.10}$$

We deduce

$$e^{\lambda_k t}=\alpha_{n-1}(t)\lambda_k^{n-1}+\ldots+\alpha_1(t)\lambda_k+\alpha_0(t) \tag{3.11}$$

The coefficients $(\alpha_0,\alpha_1,\ldots,\alpha_{n-1})$ are then found by solving the following system of n equations with n unknowns:

$$\begin{cases} e^{\lambda_0 t} & =\alpha_{n-1}(t)\lambda_0^{n-1}+\ldots+\alpha_1(t)\lambda_0+\alpha_0(t) \\ e^{\lambda_1 t} & =\alpha_{n-1}(t)\lambda_1^{n-1}+\ldots+\alpha_1(t)\lambda_1+\alpha_0(t) \\ \vdots & \\ e^{\lambda_{n-1} t} & =\alpha_{n-1}(t)\lambda_{n-1}^{n-1}+\ldots+\alpha_1(t)\lambda_{n-1}+\alpha_0(t) \end{cases} \tag{3.12}$$

<u>Example</u>: Consider the matrix $A=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}$, let's calculate e^{At} using the Cayley-Hamilton theorem.

The eigenvalues of A are $(\lambda_1=2)\ et\ (\lambda_2=-1)$. The corresponding eigenvectors Can be found by solving the homogeneous system of equations $(A-\lambda I)X=0$ for each eigenvalue. For example, for $\lambda_1=2$, we have:

$$(A-2I)X=\begin{bmatrix}0\\0\end{bmatrix}\rightarrow\begin{bmatrix}0 & 0\\0 & -3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

So $x_2 = 0$, and x_1 can be any non-zero real number. Let's take $v_1=\begin{bmatrix}1\\0\end{bmatrix}$.

In the same way, for $(\lambda_2=-1)$, the corresponding eigenvector is the solution of the characteristic equation $(A-\lambda_2 I)X=(A+I)X=0$, which gives $v_2=\begin{bmatrix}0\\1\end{bmatrix}$.

The coefficients $(\alpha_0,\alpha_1,\ldots,\alpha_{n-1})$ are the solution of:

$$\begin{cases} e^{2t} & =\alpha_1(t)\cdot 2+\alpha_0(t)\\ e^{-t} & =\alpha_1(t)\cdot(-1)+\alpha_0(t)\end{cases}$$

To solve this system of equations, let's rewrite the second equation as follows:
$\alpha_0(t)=e^{-t}+\alpha_1(t)$

Let's replace this expression in the first equation: $e^{2t}=\alpha_1(t)\cdot 2+(e^{-t}+\alpha_1(t))$

Where $e^{2t}-e^{-t}=\alpha_1(t)\cdot 2+\alpha_1(t)\rightarrow e^{2t}-e^{-t}=\alpha_1(t)\cdot 3$

And finally $\alpha_1(t)=\dfrac{e^{2t}-e^{-t}}{3}$ et $\alpha_0(t)=e^{-t}+\dfrac{e^{2t}-e^{-t}}{3}$

So : $e^{At}=\alpha_1(t)A+\alpha_0(t)I=e^{At}=\dfrac{e^{2t}-e^{-t}}{3}\cdot\begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}+\left(e^{-t}+\dfrac{e^{2t}-e^{-t}}{3}\right)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

Finally:

$$e^{At}=\frac{1}{3}\begin{bmatrix}2e^{2t}-2e^{-t}+3e^{-t}+e^{2t}-e^{-t} & 0\\0 & -e^{2t}+e^{-t}+3e^{-t}+e^{2t}-e^{-t}\end{bmatrix}=\frac{1}{3}\begin{bmatrix}3e^{2t}+e^{-t} & 0\\0 & e^{2t}+3e^{-t}\end{bmatrix}$$

III.4 Stability of systems represented in state space

The solution of the state equation shows that for the system to be stable (convergent), the transition matrix e^{At} must be convergent $\lim_{t\to+\infty} e^{At}=0$, which implies that these eigenvalues must have negative real parts.

Example: Study the stability of the system with the state matrix $A=\begin{pmatrix}1 & 0 & 1\\0 & 2 & 1\\0 & 0 & 3\end{pmatrix}$

Let's calculate its eigenvalues, they are the roots of:

$$\left|\lambda.\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&0&1\\0&2&1\\0&0&3\end{pmatrix}\right|=0\rightarrow\left|\begin{pmatrix}\lambda-1&0&-1\\0&\lambda-2&-1\\0&0&\lambda-3\end{pmatrix}\right|=0$$

$$\rightarrow(\lambda-1)\begin{vmatrix}(\lambda-2)&-1\\0&(\lambda-3)\end{vmatrix}-(-1)\begin{vmatrix}0&(\lambda-2)\\0&0\end{vmatrix}=(\lambda-1)(\lambda-2)(\lambda-3)=0$$

So the eigenvalues of A are: $\lambda_1=1$, $\lambda_2= 2$, $\lambda_3=3$. They are all strictly positive, so the system is unstable.

III.5 Notions of controllability and observability

Controllability and observability are fundamental concepts in the field of control theory for dynamic systems. They are used to evaluate the possibilities of controlling and/or measuring a system, respectively.

III.5.1Controllability

The controllability of a dynamic system describes the ability to drive the system from an initial state to a final state using a certain control law. A system is said to be completely controllable if it is possible to reach any state of the system by manipulating the appropriate inputs (commands) over a certain period of time.

Mathematically, this can be expressed by the existence of a control law $u(t)$ such that, for an initial state $x(t_0) = x_0$, the system reaches the state $(x(t_1) = x_1)$ within the time interval $[t_0\ t_1]$. This condition of controllability is fundamental in the design of control systems, as it ensures complete influence over the system's evolution by adjusting the controls.

III.5.1 Controllability Criterion

Let's consider a system represented by the state equation:

$$\begin{aligned}\dot{x}(t)&=Ax(t)+Bu(t)\\y&=Cx(t)\end{aligned}\qquad A\in\mathbb{R}^{n\times n},B\in\mathbb{R}^{n\times m},C\in\mathbb{R}^{r\times m}$$

This system is said to be **controllable** if and only if the controllability matrix:

$$C_{A,B}=(B|\ AB|...|A^{n-1}B) \qquad (3.13)$$

is of rank n (full rank).

Reminder: The rank of a matrix A corresponds to the dimension of the largest non-zero minor (square subset).

For example, the rank of the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ is $r = 2$, because its determinant is zero, and its largest non-zero minor is of dimension 2x2.

III.5.2 Observability

Observability is the ability to determine the current state of the system by analyzing its measured outputs. It evaluates the possibility of reconstructing the system's state based on the available information. A system is considered completely observable when each state can be reconstructed from the measured outputs over a given period of time. In other words, complete observability means that the current state of the system can be retroactively reconstructed from the command and output information over a given time interval.

III.5.3 Observability Criterion

A dynamic system is said to be completely observable if and only if the observability matrix:

$$\mathrm{O}_{C,A} = \begin{bmatrix} C \\ CA \\ . \\ . \\ . \\ CA^{n-1} \end{bmatrix} \tag{3.14}$$

is of rank n (full rank).

Example :

Let's study the observability of the system whose state matrices are:

$A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

The observability matrix is $O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

The rank of O is 2 (its determinant is non-zero), so the system is completely observable.

Note:

- ✓ The canonical form of controllability is controllable.
- ✓ The canonical form of observability is observable.
- ✓ Duality of controllability-observability:

A system with state matrices $(A,\ B, C, D)$ is controllable if and only if its dual $(A^T,\ C^T, B^T, D)$ is observable, and vise versa.

Example :

Let's consider another example of a dynamic system whose state matrices are:

$$A=\begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}, B=\begin{bmatrix} 1 \\ 1 \end{bmatrix}, C=\begin{bmatrix} 1 & 0 \end{bmatrix}$$

Let's verify the duality Let's verify the duality of controllability-observability by considering the pair $(A,\ B)$ and the pair (B^T, A^T) .

- ✓ The controllability matrix is $C=\begin{bmatrix} B & AB \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}$, the rank of C is 2, so the system is controllable.
- ✓ The associated Observability matrix (B^T, A^T) is : $O=\begin{bmatrix} B^T \\ A^T B^T \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$, the rank of O is 2 , so the pair (B^T, A^T) is observable.