Chapter 4. State feedback control

IV.1 Introduction

State feedback control is a method used in feedback control systems where the state vector (which represents the internal state of the system) is used to calculate the control action. This approach allows for the modification of the system's behavior based on its state representation. However, it is only applicable if the system is controllable.

IV.2 State feedback control law

The following figure shows the state feedback loop structure:

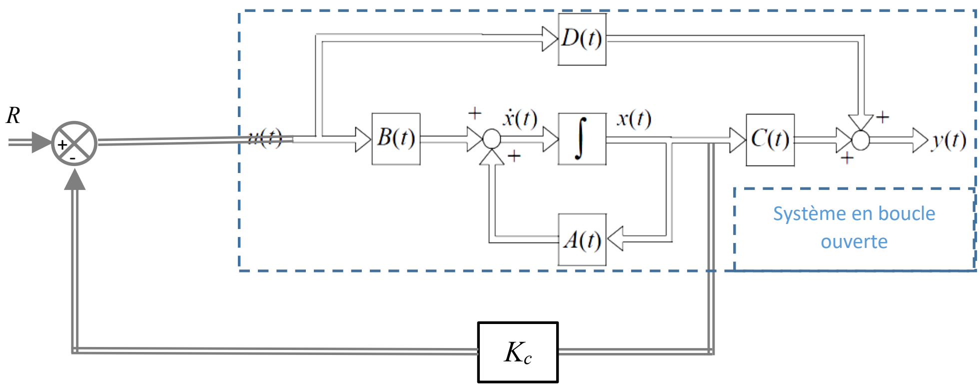

Fig 4.1 Looping by state return

The state representation of the open-loop system is:

$$\begin{cases} \dot{x} = Ax + B.u \\ y = Cx + D.u \end{cases} \tag{0.1}$$

The state feedback control law is given by:

$$u = -K_c x + R \tag{0.2}$$

By substituting u into the equation of the open-loop system, we obtain the state representation of the closed-loop system (corrected):

$$\begin{cases} \dot{x} = \left(A - BK_c\right)x + B.R \\ y = \left(C - DK_c\right)x + D.R \end{cases} \tag{0.3}$$

The eigenvalues (or poles) of the new system are then the roots of the characteristic polynomial:

$$P_{(A-BK_c)}\left(s\right) = \det\left(sI - \left(A - BK_c\right)\right) \tag{0.4}$$

If the system is controllable, then it is possible to perform pole placement in the complex plane. This placement becomes simpler by considering the canonical form of controllability:

$$A_c - B_cK_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_0 - k_1^c & -a_1 - k_2^c & \cdots & \cdots & -a_{n-1} - k_n^c \end{bmatrix} \tag{4.5}$$

With A_c, B_c the state matrices of the system in the canonical form of controllability.

The coefficients of the state feedback vector K_c can be obtained by the pole placement method, by comparing the characteristic polynomial of the controllable form in closed loop with the polynomial formed by the desired closed-loop poles as follows:

$$\begin{aligned} &P_{A_c - B_cK_c}(s) = s^n + (a_{n-1} + k_n^c)s^{n-1} + \ldots + (a_1 + k_2^c)s + (a_0 + k_1^c) \\ &\textit{polynôme désiré } P_d(s) = s^n + a_{n-1}^d s^{n-1} + \ldots + a_1^d s + a_0^d \end{aligned} \tag{4.6}$$

The K_C coefficients are determined by identification as follows:

$$k_i^c = a_{i-1}^d - a_{i-1} \qquad 1 \le i \le n \tag{4.7}$$

This formula is applicable to any controllable system put into its canonical controllability form. In the following section, we will see how to transform a controllable system into its canonical controllability form.

IV.2.1 Steps for implementing the pole placement method

To implement the pole placement method in state feedback control for a system in the canonical controllable form, follow these steps:

1) If the system is in controllable form: combine the state equations of the system with the feedback terms k_i^c to create the characteristic equation of the closed-loop system: $P_{(A-BK_c)}(s) = \det\left(sI - (A - BK_c)\right)$
2) Establish the desired characteristic equation from the desired poles: $P_d(s) = (s-p_1)(s-p_2)...(s-p_n) = s^n + a_{n-1}^d s^{n-1} + ... + a_1^d s + a_0^d$
3) Equate the two characteristic equations $P_{(A-BK_c)}(s) = P_d(s)$, and determine the gains k_i^c by term-by-term identification.

Example 1 :

Let the state matrices of a system be in the canonical form of controllability:

$$A_c = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We want the closed-loop system to have the poles (-2, -3). The desired characteristic polynomial is therefore expressed as follows $P_d(s) = (s + 2)(s + 3) = s^2 + 5s + 6$: .

Let's apply formula (4.7):

$$\begin{cases} k_1^c = a_0^d - a_0 = 6 - 3 = 3 \\ k_2^c = a_1^d - a_1 = 5 - 4 = 1 \end{cases}$$

So : $K_c = [3 \quad 1]$.

If the system is not in the canonical form of controllability, it must first be put into this form by transforming it. The following section presents the method for transitioning to the canonical form of controllability.

IV.2.2 Transition to the canonical controllability form

We have seen that the canonical controllability form is more suitable for state feedback control, hence the need to use a method to transform any controllable system into a canonical controllability form.
For any controllable system, there exists a regular change of basis T that allows any system to be put into the canonical controllability form: $z = Tx$. The state representation of the transformed system then becomes:

$$\begin{cases} \dot{z} = TAT^{-1}z + TBu \\ y = CT^{-1}z + Du \end{cases} \tag{4.8}$$

With T transformation matrix expressed by:

$$T = (C_{A_c,B_c})(C_{A,B})^{-1} \tag{4.9}$$

With :

$C_{A,B} = (B|\ AB|...|A^{n-1}B\ )$ controllability matrix of the system,

$C_{A_c,B_c} = (B_c|\ A_cB_c|...|A_c^{n-1}B_c)$ controllability matrix of the system put in the canonical controllability form,

$$A_c = \begin{bmatrix} 0 & 1 & 0 & ... & ... & 0 \\ 0 & 0 & 1 & 0 & ... & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & & & & 0 & 1 \\ -a_0^c & -a_1^c & \cdots & \cdots & -a_{n-2}^c & -a_{n-1}^c \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$ state matrices of the system in the canonical controllability form.

Example 2 :
Let's consider the state representation of a dynamic system:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}, \text{ with : } A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

The state matrices of the controllability canonical form of this representation are:

$$A_c = \begin{bmatrix} 0 & 1 \\ -a_0^c & -a_1^c \end{bmatrix}, B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

a_0^c and a_1^c are the coefficients of the characteristic equation: $\det(sI - A) = 0$.

We have : $\det(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}) = \begin{bmatrix} s-1 & -2 \\ -3 & s \end{bmatrix} = s(s-1) - 6 = s^2 - s - 6$, so $a_0^c = -6, a_1^c = -1$

Finally $A_c = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}$.

Next, we construct the transformation matrix as follows:

$$T = (C_{A_c,B_c})(C_{A,B})^{-1} = (B_c, A_c B_c)(B, AB)^{-1}$$

We have $(B, AB)^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$, and $(C_{A_c,B_c}) = (B_c, A_c B_c) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, so:

$$T = \frac{1}{3}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} \text{ and } T^{-1} = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$$

Now, let's perform the change of variable $\left(z = Tx\right)$, the state equation in its controllable canonical form then becomes:

$$\begin{cases} \dot{z} = TAT^{-1}z + TBu \\ y = CT^{-1}z + Du \end{cases} \rightarrow \begin{cases} \dot{z} = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} z \end{cases}$$

Now, the system is in controllable form, ready for state feedback controller design.

IV.3 Ackermann's Formula

The Ackermann formula is a method used in state feedback control to calculate the gain matrix *K* directly from the gain matrix *Kc* obtained from the controllability form of the system. For a controllable system, Ackermann's formula is expressed as follows:

$$K = K_c T^{-1} \tag{4.10}$$

With :

- K is the gain matrix of the system,

- K_c is the gain matrix calculated from the canonical form of controllability,

- T is the transformation matrix associated with the canonical form.

Example 3 :

Let's revisit the system from Example 2 where:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

Let's apply Ackermann's formula to find the state feedback gain to have the desired poles (-1, -2) in closed loop. First, let's calculate the feedback gain of the system in the canonical controllability form K_c.

The desired characteristic polynomial is expressed as follows:

$P_d(s) = (s + 1)(s + 2) = s^2 + 3s + 2$.

We have already calculated $A_c = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}$ in the previous example, so let's directly apply formula (4.7):

$$\begin{cases} k_1^c = a_0^d - a_0 = 2 - (-6) = 8 \\ k_2^c = a_1^d - a_1 = 3 - (-1) = 4 \end{cases}$$

So : $K_c = \begin{bmatrix} 8 & 4 \end{bmatrix}$.

Finally, we have: $K = K_c T^{-1} = \begin{bmatrix} 8 & 4 \end{bmatrix} \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix} = \begin{bmatrix} 12 & 8 \end{bmatrix}$.