CHAPTER II STATE REPRESENTATION OF SYSTEMS

II.1 Introduction

There are several forms of mathematical representation of systems: transfer functions in the frequency domain, differential equations in the time domain, and state representation, also in the time domain. This last one is an essential approach in modern automation for modeling the dynamic behavior of systems, using first-order differential equations, describing the evolution of state variables over time. This method is particularly useful for analyzing and designing complex systems such as control systems and electronic circuits.

II.2 Notion of state

We define the state of a system at time as the information about its past (history), necessary to know its evolution if we know the input signals and the system's equations.

II.3 Variables d'état State variables

It is the set of variables that are capable of memorizing information about the system's dynamics. Generally, we denote them as: $x_1,\ x_2,\ x_3,\ldots$

A process described in the form of a state representation is defined by its state variables, which embody energy parameters. These variables provide a detailed description of the internal evolution of the system.

II.4 State vectors

It is a minimal set of state variables, necessary and sufficient to determine the evolution of a system if we know the equations that describe its operation and its input signals. Generally, we denote them as:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \end{bmatrix} \tag{2.1}$$

It is important to note that state vectors are not unique. Indeed, they differ depending on the choice of state variables.

II.5 Equation of state

Any linear, causal, and continuous system can be represented by the following matrix equations:

$$\begin{aligned} \dot{X}(t) &= A(t)X(t) + B(t)e(t) \qquad \textit{Equation d'état} \\ Y(t) &= C(t)X(t) + D(t)e(t) \qquad \textit{Equation de mesure (ou de sortie)} \end{aligned} \tag{2.2}$$

- *A, B, C, D* are called system state matrices.
- *X* is called the state vector of the system.
- *e* is called the input vector of the system.
- *Y* is called the output vector of the system.

Note –

- If the matrices A, B, C, and D are time-independent (constant), then the system is stationary.
- In the case of a discrete system, these equations take the following form: In the case of a discrete system, these equations take the following form:

$$\begin{aligned} X(k+1) &= AX(k) + Be(k) \\ Y(k) &= CX(k) + De(k) \end{aligned} \tag{2.3}$$

Example

In order to illustrate the state representation, consider the following electrical diagram:

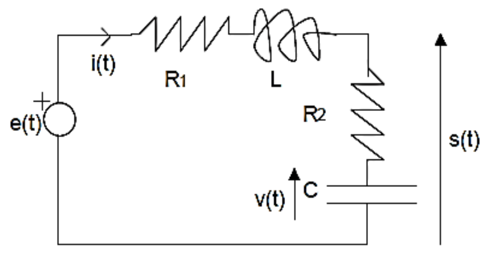

Fig 2.1. Electrical circuit

His electrical equations are given by:

$$\begin{cases} e(t) = (R_1 + R_2)i(t) + L\dfrac{di(t)}{dt} + v(t) \\ i(t) = C\dfrac{dv(t)}{dt} \\ s(t) = R_2 i(t) + v(t) \end{cases}$$

If we choose *i*(t) and *v*(t) as state variables, *e(t)* as input, and *S(t)* as output, then we can write:

$$\frac{dX}{dt} = \begin{bmatrix} \dfrac{-(R_1 + R_2)}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & 0 \end{bmatrix} X + \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix} e$$

$$Y = \begin{bmatrix} R_2 & 1 \end{bmatrix} X$$

$$\text{With : } X = \begin{bmatrix} i(t) \\ v(t) \end{bmatrix}, Y = S(t)$$

II.6 Transition from the state equation to the transfer function

Let the state and measurement equations of a system be:

$$\dot{X}(t) = AX(t) + Be(t)$$
$$Y(t) = CX(t) + De(t)$$

Let's apply the Laplace transform, we find:

$$sX(s) = AX(s) + BE(s)$$
$$Y(s) = CX(s) + DE(s)$$

Assuming that the initial conditions are zero, we can write:

$$X(s) = X(s)(sI - A) = BE(s) \rightarrow X(s) = (sI - A)^{-1} BE(s)$$
$$Y(s) = CX(s) + DE(s) = C(sI - A)^{-1} BE(s) + DE(s) = \left[C(sI - A)^{-1} B + D \right] E(s)$$

With I identity matrix of the same dimension as A

We then deduce the transfer function of the system as follows:

$$G(s)=\frac{S(s)}{E(s)}=C(s.I-A)^{-1}.B+D \tag{2.4}$$

Important note:
The poles of a transfer function are the eigenvalues of the matrix A

II.7 Canonical Forms of State Representations

Let's consider a system represented by its transfer function (with m < n):

$$G(s)=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_1 s^1+b_0}{s^n+a_{n-1}s^{n-1}+\ldots+a_1 s^1+a_0} \tag{2.5}$$

The state representation of this system is not unique; there are several canonical forms listed below.

II.7.1 Commandable form (companion for the command)

This configuration is interpreted as a series of pure integrators, whose outputs represent the state variables of the system. Let the general form of a system's transfer function be:

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_1 s^1+b_0}{s^n+a_{n-1}s^{n-1}+\ldots+a_1 s^1+a_0}\rightarrow$$

$$\frac{d^n y(t)}{dt^n}+\ldots+a_1\frac{dy(t)}{dt}+a_0 y(t)=b_m\frac{d^m u(t)}{dt^m}+\ldots+b_1\frac{du(t)}{dt}+b_0 u(t)$$

The following figure illustrates the block diagram of this configuration.

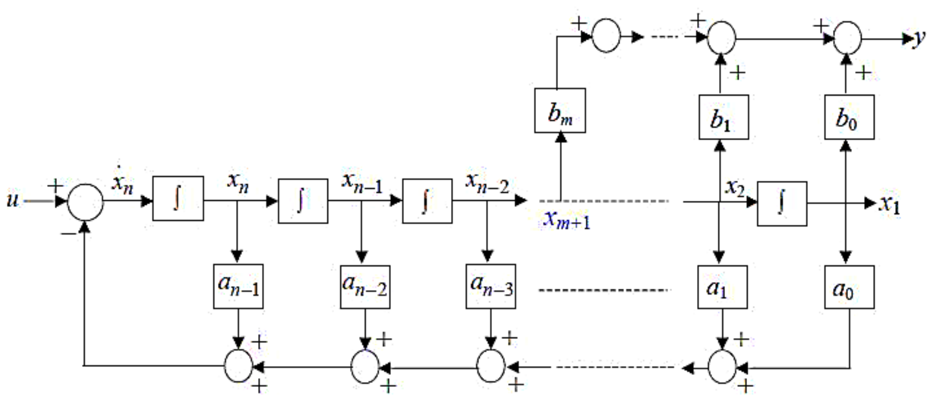

Fig 2.2 Companion order form

a) Case of simple excitation

Let's first take the case of simple excitation:

$$G(s)=\frac{Y(s)}{U(s)}=\frac{1}{s^n+a_{n-1}s^{n-1}+\ldots+a_1s^1+a_0}\to$$

$$\frac{d^n y(t)}{dt^n}+\frac{d^{n-1}y(t)}{dt^{n-1}}+\ldots+a_1\frac{dy(t)}{dt}+a_0y(t)=u(t)\to$$

$$\frac{d^n y(t)}{dt^n}=-\frac{d^{n-1}y(t)}{dt^{n-1}}\ldots-a_1\frac{dy(t)}{dt}-a_0y(t)+u(t)$$

If we choose the state variables as follows:

$$x_1(t)=y(t),x_2(t)=\frac{dy(t)}{dt},\ldots,x_n(t)=\frac{d^n y(t)}{dt^n}$$

So we can write:

$\dot{x}_1(t)=x_2(t),\dot{x}_2(t)=x_3(t),\ldots,x_{n-1}(t)=x_n(t)$, and finally :

$$\dot{x}_n(t)=-a_0x_1(t)-a_1x_2(t)-\ldots-a_{n-1}x_n(t)+u(t)$$

Hence the state equations of the controllable form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_{n-1}\\ \dot{x}_n\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & \cdots & \cdots & 0\\ 0 & 0 & 1 & 0 & \cdots & 0\\ \vdots & & \ddots & \ddots & & \vdots\\ \vdots & & & \ddots & \ddots & 0\\ 0 & & & & 0 & 1\\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-2} & -a_{n-1}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \vdots\\ 0\\ 1\end{bmatrix}u(t) \tag{2.6}$$

$$y(t)=\begin{bmatrix}1 & 0 & \cdots & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix}$$

Remarks: In this configuration, each state variable is the derivative of the previous one, which means that if we modify the command *u(t)*, all the states are changed. For this reason, we refer to it as a controllable system.

The analog diagram of this configuration is illustrated by the following figure:

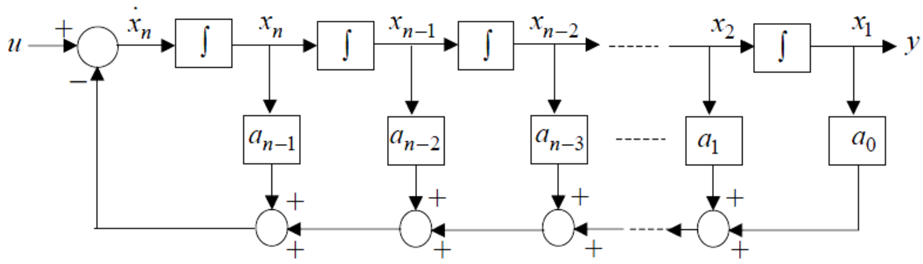

Fig 2.3 Analog diagram (controllability form in the case of simple excitation)

b) Case of multiple excitation

Let $G(s)=\dfrac{Y(s)}{U(s)}=\dfrac{b_m s^m+b_{m-1}s^{m-1}+\dots+b_1 s^1+b_0}{s^n+a_{n-1}s^{n-1}+\dots+a_1 s^1+a_0}$, $m<n$ and $b_0\neq 0$

Let's introduce an intermediate variable V such that: $G(s)=\dfrac{Y(s)}{V(s)}\dfrac{V(s)}{U(s)}$ with

$$\frac{V(s)}{U(s)}=\frac{1}{s^n+a_{n-1}s^{n-1}+\dots+a_1 s^1+a_0} \quad et \quad \frac{Y(s)}{V(s)}=b_m s^m+b_{m-1}s^{m-1}+\dots+b_1 s^1+b_0$$

We notice that $\dfrac{V(s)}{U(s)}$ corresponds to the previous case (simple excitation), so if we set:

$$x_1(t)=v(t), x_2(t)=\frac{dv(t)}{dt},\dots,x_n(t)=\frac{d^n v(t)}{dt^n}$$

We will have:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_{n-1}\\ \dot{x}_n\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & \cdots & \cdots & 0\\ 0 & 0 & 1 & 0 & \cdots & 0\\ \vdots & & \ddots & \ddots & & \vdots\\ \vdots & & & \ddots & \ddots & 0\\ 0 & & & & 0 & 1\\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-2} & -a_{n-1}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix}+\begin{bmatrix}0\\0\\ \vdots\\0\\1\end{bmatrix}u(t)$$

Whereas:

$$\frac{Y(s)}{V(s)} = b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s^1 + b_0 \to Y(s) = V(s)(b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s^1 + b_0) \to$$

$$y(t) = b_m \frac{dv^m}{dt^m} + b_{m-1} \frac{dv^{m-1}}{dt^{m-1}} + \ldots + b_1 \frac{dv}{dt} + b_0 v = b_m x_{m+1} + b_{m-1} x_{m+1} + \ldots + b_1 x_2 + b_0 x_1$$

Hence the state representation :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & & & & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} b_0 & b_1 & \cdots & b_m & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m+1} \\ x_{m+2} \\ \vdots \\ x_n \end{bmatrix} \tag{2.7}$$

The simulation diagram in the case of multiple excitation is represented by the following figure:

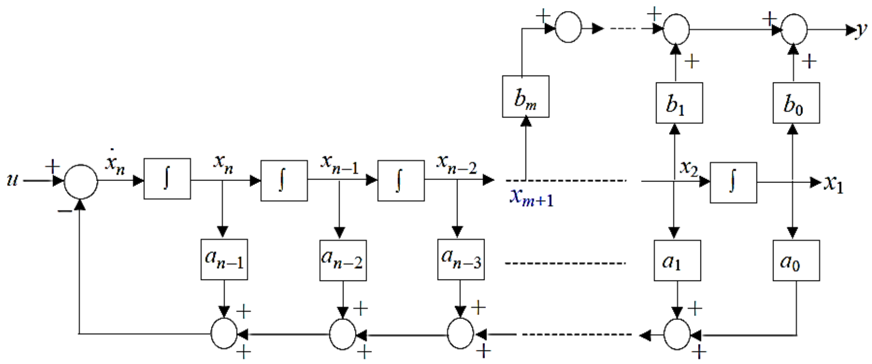

Fig 2.4 Simulation diagram of the controllable form "multiple excitation case"

II.7.2 Observable form (companion for observation)

Let: $G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s^1 + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s^1 + a_0}$, with : $m < n$. We will have:

$$\frac{d^n y(t)}{dt^n}+\frac{d^{n-1} y(t)}{dt^{n-1}}+\ldots+a_1\frac{dy(t)}{dt}+a_0 y(t)=b_m\frac{d^m u(t)}{dt^m}+b_{m-1}\frac{d^{m-1} u(t)}{dt^{m-1}}+\ldots+b_1\frac{du(t)}{dt}+b_0 u(t)$$

The Laplace transform of this equation gives:

$$s^nY(s)+a_{n-1}s^{n-1}Y(s)+\ldots+a_1sY(s)+a_0Y(s)=b_ms^mU(s)+b_{m-1}s^{m-1}U(s)+\ldots+b_1sU(s)+b_0U(s)\rightarrow$$
$$s^nY(s)=\left[b_0U(s)-a_0Y(s)\right]+s\left[b_1U(s)-a_1Y(s)\right]+\ldots+s^m\left[b_mU(s)-a_mY(s)\right]-\ldots-a_{n-1}s^{n-1}Y(s)$$

By dividing by s^n, we get:

$$Y(s)=\frac{\left[b_0U(s)-a_0Y(s)\right]}{s^n}+\frac{\left[b_1U(s)-a_1Y(s)\right]}{s^{n-1}}+\ldots+\frac{\left[b_mU(s)-a_mY(s)\right]}{s^{n-m}}-\ldots-\frac{a_{n-1}Y(s)}{s}$$

If we choose:

$$X_n(s)=Y(s)\rightarrow x_n(t)=y(t)$$
$$X_1=\frac{\left[-a_0Y(s)+b_0U(s)\right]}{s}\rightarrow \dot{x}_1=-a_0x_n+b_0U(s)$$
$$X_2=\frac{\left[-a_1Y(s)+b_1U(s)+x_1\right]}{s}\rightarrow \dot{x}_2=x_1-a_1x_n+b_1u$$
$$\vdots$$
$$X_{m+1}=\frac{\left[-a_mY(s)+b_mU(s)+x_m\right]}{s}\rightarrow \dot{x}_{m+1}=x_m-a_mx_n+b_mu$$
$$X_{m+2}=\frac{\left[-a_{m+1}Y(s)+x_{m+1}\right]}{s}\rightarrow \dot{x}_{m+2}=x_{m+1}-a_{m+1}x_n$$
$$\vdots$$
$$X_n=\frac{\left[-a_{n-1}Y(s)+x_{n-11}\right]}{s}\rightarrow \dot{x}_n=x_{n-1}-a_{n-1}x_n$$

Hence the state equations of the observable form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{m+1} \\ \vdots \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & -a_m \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \\ \vdots \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & \cdots & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

In the case where $(m = n)$, the transfer function decomposes into a direct transmission and a part where $(m < n)$. In this case, the matrix D will be equal to the direct transmission, and the part where $(m < n)$ will be treated as before.

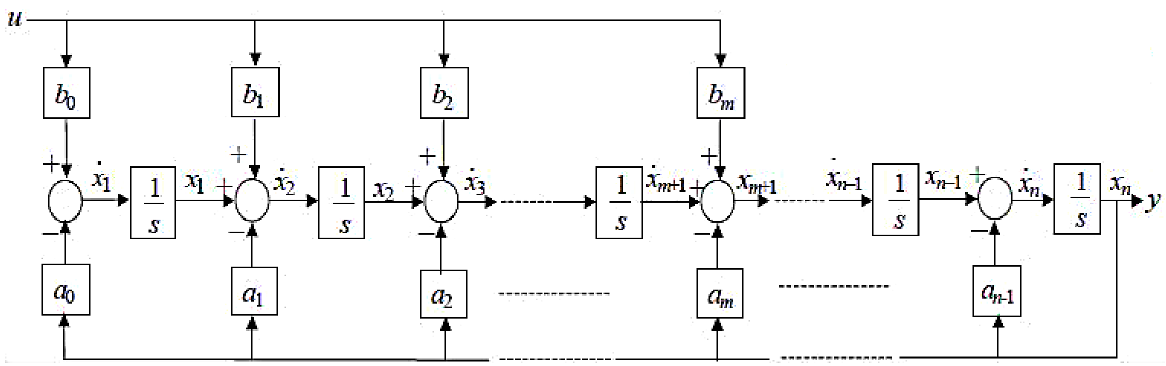

Fig 2.5 Simulation scheme of the observable form

Note:

It is observed that the two canonical forms of controllability and observability are symmetric: one is obtained by transposing the other with respect to the main diagonal.

Indeed: let be $(A_c,\ B_c,\ C_c,\ D_c)$ the canonical controllability $(A_o,\ B_o,\ C_o,\ D_o)$ realization and be the canonical observability realization. We observe that:

$$(A_c, B_c, C_c, D_c) = (A_o^T, B_o^T, C_o^T, D_o^T)$$

This relationship highlights the duality between the canonical forms of controllability and observability, where each element of the canonical realization of controllability is the transpose of that of observability.

II.7.3 Modal Form

In this form, if the transfer function ($m < n$) has simple poles, it would be possible to decompose it into simple elements as follows:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s^1 + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s^1 + a_0} = \frac{\alpha_1}{s+\lambda_1} + \frac{\alpha_2}{s+\lambda_2} + \ldots + \frac{\alpha_n}{s+\lambda_n} \rightarrow$$

$$Y(s) = \frac{\alpha_1}{s+\lambda_1} U(s) + \frac{\alpha_2}{s+\lambda_2} U(s) + \ldots + \frac{\alpha_n}{s+\lambda_n} U(s)$$

With: the λ_i are the eigenvalues of the matrix A (poles of $G(s)$).

Let's assume: $X_1(s) = \frac{U(s)}{s+\lambda_1}, X_2(s) = \frac{U(s)}{s+\lambda_2}, \ldots, X_n(s) = \frac{U(s)}{s+\lambda_n}$, we will have:

$$\dot{x}_1(t) = -\lambda_1 x_1(t) + u(t)$$
$$\dot{x}_2(t) = -\lambda_2 x_2(t) + u(t)$$
$$\ldots$$
$$\dot{x}_n(t) = -\lambda_n x_n(t) + u(t)$$

$$y(t) = \alpha_1 x_1(t) + \ldots + \alpha_{n-1} x_n(t)$$

Hence the state representation:

$$\dot{X}(t) = \begin{bmatrix} -\lambda_0 & 0 & \ldots & 0 & 0 \\ 0 & -\lambda_1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & -\lambda_{n-1} \end{bmatrix} X(t) + \begin{bmatrix} 1 \\ 1 \\ \ldots \\ 1 \end{bmatrix} u(t) \tag{2.6}$$
$$y(t) = \begin{bmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{bmatrix} X(t)$$

The simulation diagram of the modal form is shown in the following figure:

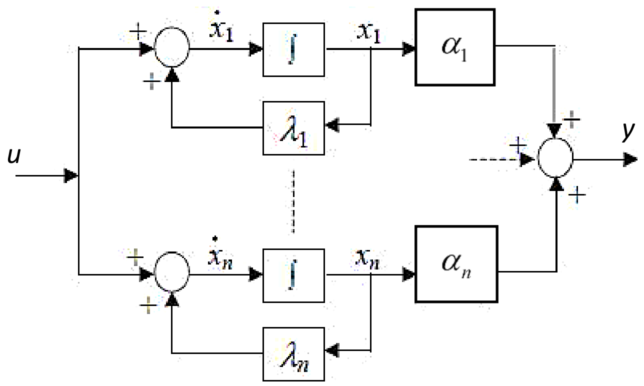

Fig 2.5 Simulation diagram of the modal shape

Note:

- If the system has simple poles, then the matrix A is diagonal
- In the case $(m = n)$, there will be a direct transmission α_0, and the matrix $D = \alpha_0$ in the state representation.

❖ Case of a multiple pole

Let there be a transfer function $G(s) = \dfrac{Y(s)}{U(s)}$ possessing a multiple real pole of order q as well as simple poles, we have:

$$Y(s) = \left[\frac{r_1}{(s-\lambda_1)^q} + \frac{r_2}{(s-\lambda_1)^{q-1}} + \ldots \frac{r_q}{s-\lambda_1} + \sum_{i=q+1}^{n} \frac{r_i}{s-\lambda_i} + r_0 \right] U(s)$$

If we choose the state variables as follows:

- For the multiple pole : $X_1 = \dfrac{1}{(s-\lambda_1)^q} U; X_2 = \dfrac{1}{(s-\lambda_1)^{q-1}};\ldots; X_q = \dfrac{1}{s-\lambda_1}$

So : $X_1 = \dfrac{1}{s-\lambda_1} X_2; \quad X_2 = \dfrac{1}{s-\lambda_1} X_3; \quad \ldots; \quad X_{q-1} = \dfrac{1}{s-\lambda_1} X_q; X_q = \dfrac{1}{s-\lambda_1} U$

- For simple poles : $X_i = \dfrac{1}{s-\lambda_i} U; \quad i = q+1, \ldots, n$

So : $X_i = \dfrac{1}{s-\lambda_1} X_{i-1}$

By applying the inverse Laplace transform, we will have:

$$
\begin{aligned}
\dot{x}_1 &= \lambda_1 x_1 + x_2 \\
\dot{x}_2 &= \lambda_1 x_2 + x_3 \\
&\ldots \\
\dot{x}_q &= \lambda_1 x_q + u \\
\dot{x}_i &= \lambda_i x_i + u; \quad i = q+1, \ldots \\
y &= \sum_{i=1}^{n} r_i x_i + r_0 u
\end{aligned}
$$

The state representation will then be as follows:

Bloc de Jordan

$$
\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_{k-1} \\ \dot{x}_k \\ \dot{x}_{k+1} \\ \vdots \\ \dot{x}_n \end{bmatrix}
=
\left[\begin{array}{cccc|ccc}
\lambda_1 & 1 & \cdots & 0 & & & \\
0 & \ddots & \ddots & \vdots & & 0 & \\
\vdots & \ddots & \lambda_1 & 1 & & & \\
0 & \cdots & 0 & \lambda_1 & & & \\
\hline
 & & & & \lambda_{k+1} & & \\
 & 0 & & & & \ddots & \\
 & & & & & & \lambda_n
\end{array}\right]
\begin{bmatrix} x_1 \\ \vdots \\ x_{k-1} \\ x_k \\ x_{k+1} \\ \vdots \\ x_n \end{bmatrix}
+
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(t)
\qquad (2.10)
$$

$$
y(t) = [r_1 \ldots\ldots\ldots r_n] \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + r_0 u
$$

Application exercises:

1) Let the transfer function of a system $G(s) = \dfrac{2s+1}{s^3 + 2s^2 - s - 2}$, its state representation in the controllable form will have the following state matrices:

$$
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; C = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}
$$

Question : Give its state representation in observable form

2) Let the transfer function of a system be $G(s) = \frac{1}{(s+1).(s+2)^2}$, its state representation in Jordan's modal form will have the following state matrices

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}; B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

<u>Question :</u> Give the vector C