L1 Informatics
Algebra 1

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# Series of Exercises 01
# Mathematical Logic and Proofs

**Exercice 1** *Among the following expressions, which ones are propositions ? For propositions, indicate whether they are true or false.*

1. 11 *is a prime number.*
2. $((-5)^2 = 25) \quad \wedge \quad (\sqrt{25} = -5)$.
3. $(\sqrt{3}$ *is an integer*$) \quad \vee \quad ($*all the sides of a square are equal*$)$.
4. $\forall x \in \mathbb{R}, \quad x + 2 > 5$.
5. $\exists z \in \mathbb{C}, \quad |z| = 1$.
6. $x \in \mathbb{N}$.

**Exercice 2** *Let $P, Q$ and $R$ be propositions. Using truth table, show the follwing relations :*

1. $\overline{(P \wedge Q)} \Leftrightarrow (\bar{P} \vee \bar{Q})$
2. $\overline{(P \Rightarrow Q)} \Leftrightarrow (P \wedge \bar{Q})$ ***(homework)***
3. $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge Q)$
4. $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee Q)$ ***(homework)***.

**Exercice 3** *Consider the following four propositions :*

1. $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : x + y > 0$.
2. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x + y > 0$.
3. $\forall x \in \mathbb{R}, \forall y \in \mathbb{R} : x + y > 0$.
4. $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : y^2 > x$.
   *Are these propositions true or false ? Provide their negations.*

**Exercice 4** *Let $f$ be a function from $\mathbb{R}$ to $\mathbb{R}$, write in terms of quantifiers the following expressions :*

1. *$f$ is positive.*
2. *$f$ is even.*
3. *$f$ is strictly increasing function.*
4. *$f$ is bounded above by a constant $M$.*

**Exercice 5**

1. *Let $a$ and $b$ be two real numbers. Using direct reasoning, show that :*
$$a^2 + b^2 = 1 \Rightarrow |a + b| \leq 2$$
2. *Let $n \in \mathbb{N}$, by contrapositive, Prove that :*
$$n^2 \text{ is divisible by } 3 \Rightarrow n \text{ is divisible by } 3.$$
3. *Using the proof by contradiction prove that :*
   a) $\sqrt{3}$ *is an irrational number.*
   b) $\forall x \in \mathbb{R}^* : \quad \sqrt{1 + x^2} \neq 1 + \frac{x^2}{2}$.***(homework)***
4. *Prove by induction :*
   a) $\forall n \in \mathbb{N}^*, 1 + 2^3 + 3^3 + \ldots + n^3 = \dfrac{n^2(n+1)^2}{4}$
   b) $\forall n \in \mathbb{N} : (21^n - 2^{2n})$ *is divisible by* 17. ***(homework)***

# Corrected Excercise

## Excercise 1

1) 11 is a prime number.

This expression is a true proposition.

2) $((-5)^2 = 25) \wedge (\sqrt{25} = -5)$.

This expression is a false proposition because:

$$\sqrt{25} \neq -5$$

3) ($\sqrt{3}$ is an integer) $\vee$ (all the sides of a square are equal).

This expression is a true proposition because:

all the sides of a square are equal (True)

4) $\forall x \in \mathbb{R}, \; x+2 > 5$.

This expression is a false proposition because:

for $x = 1 \in \mathbb{R}$, we have $x+2 = 3 < 5$

5) $\exists z \in \mathbb{C}, \; |z| = 1$.

This expression is a true proposition because:

there exists $z = i \in \mathbb{C}$ such that: $|z| = 1$.

6) $x \in \mathbb{N}$.

This expression is not proposition because we don't know the nature of the element $x$, so we cannot assign a truth value to it.

## Excercise 2.

1) $(\overline{P \wedge Q}) \Leftrightarrow (\overline{P} \vee \overline{Q})$

| $P$ | $Q$ | $P \wedge Q$ | $\overline{P \wedge Q}$ | $\overline{P}$ | $\overline{Q}$ | $\overline{P} \vee \overline{Q}$ | $(\overline{P \wedge Q}) \Leftrightarrow (\overline{P} \vee \overline{Q})$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |

3) $\underbrace{P \wedge (Q \vee R)}_{(a)} \Leftrightarrow \underbrace{(P \wedge Q) \vee (P \wedge R)}_{(b)}$

| $P$ | $Q$ | $R$ | $Q \vee R$ | (a) | $P \wedge Q$ | $P \wedge R$ | (b) | (a) $\Leftrightarrow$ (b) |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

## Exercise 3.

1) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : x + y > 0$

The proposition is false: Can we find a real number $x$ such that for every real number $y$, their sum is always positive? It is not always true, for example we can take $y = -x - 1$ we have: $x + y = x - x - 1 = -1 < 0$

The negation: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x + y \leq 0$

2) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x + y > 0$

The proposition is True: Indeed, for every real number $x$, there exists a $y$ dependent on $x$.

Let's take, for example: $y = -x + 1$

which implies that $x + y = x - x + 1 = 1 > 0$

The negation: $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : x + y \leq 0$

3) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R} : x + y > 0$

The proposition is false: we just need to find an $x$ and any $y$ that do not satisfy (3).

for example, let $x = 0$, $y = -1$

$$x + y = -1 < 0$$

The negation: $\exists x \in \mathbb{R}, \exists y \in \mathbb{R} : x + y \leq 0$

4) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : y^2 > x$

The proposition is true: there exist for example $x = -1 \in \mathbb{R}$, for all $y \in \mathbb{R}$ such that:

$$y^2 \geqslant 0 > x = -1 .$$

The negation: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : y^2 \leq x$

## Exercise 4.

1) $f$ is positive: $\forall x \in \mathbb{R}: f(x) \geq 0$

2) $f$ is even: $\forall x \in \mathbb{R}: f(-x) = f(x)$

3) $f$ is strictly increasing function:

$$\forall x_1, x_2 \in \mathbb{R}: x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$$

4) $f$ is bounded above by a constant $M$:

$$\forall x \in \mathbb{R}: f(x) \leq M$$

## Exercise 5.

1) let $a, b$ two real numbers, using direct proof, show that:

$$a^2 + b^2 = 1 \Rightarrow |a+b| \leq 2$$

We assume that $a^2+b^2=1$ and we show that: $|a+b| \leq 2$

let: $a^2+b^2 = 1 \Rightarrow \begin{cases} a^2 = 1 - b^2 \\ \text{and} \\ b^2 = 1 - a^2 \end{cases} \Rightarrow \begin{cases} 1-b^2 \geq 0 \\ \text{and} \\ 1-a^2 \geq 0 \end{cases} \Rightarrow \begin{cases} b^2 \leq 1 \\ a^2 \leq 1 \end{cases}$

then: $\begin{cases} -1 \leq a \leq 1 \\ -1 \leq b \leq 1 \end{cases} \Rightarrow -2 \leq a+b \leq 2 \Rightarrow |a+b| \leq 2$

2) let $n \in \mathbb{N}$: $n^2$ is divisible by 3 $\Rightarrow$ $n$ is divisible by 3

The Contrapositive:

$n$ is not divisible by 3 $\Longrightarrow$ $n^2$ is not divisible by 3

using direct proof we have:

we assume $n$ is not divisible by 3, then:

$n = 3k+1 \;\vee\; n = 3k+2$ for some integer $k$

Case 1: $n = 3k+1 \Rightarrow n^2 = (3k+1)^2 = 9k^2+6k+1 = 3(\underbrace{3k^2+2k}_{k'})+1$

then: $n^2 = 3k'+1$ where $k' = 3k^2+2k \in \mathbb{Z}$

So, $n^2$ is not divisible by 3.

Case 2: $n = 3k+2 \Rightarrow n^2 = (3k+2)^2 = 9k^2+12k+4 = 3(\underbrace{3k^2+4k+1}_{k'})+1$

then: $n^2 = 3k'+1$ where $k' = 3k^2+4k+1 \in \mathbb{Z}$

So, $n^2$ is not divisible by 3

In both cases: if $n$ is not divisible by 3 $\Rightarrow$ $n^2$ is not divisible by 3

Therefore, $n^2$ is divisible by 3 $\Rightarrow$ $n$ is divisible by 3

3) a) Let's prove by contradiction that: $\sqrt{3}$ is irrational

Assume that: $\sqrt{3}$ is rational, then we can write:

$$\sqrt{3} = \frac{p}{q} \quad \text{where: } \begin{cases} * \; p, q \in \mathbb{Z} \\ * \; q \neq 0 \\ * \; PGCD(p, q) = 1 \end{cases}$$

we have: $\sqrt{3} = \frac{p}{q} \Rightarrow 3 = \frac{p^2}{q^2} \Rightarrow p^2 = 3q^2$

then $p^2$ is multiple of $\Rightarrow$ $p$ is multiple of

So we can write: $p = 3k$, $k \in \mathbb{Z}$

Then: $p^2 = (3k)^2 = 9k^2 = 3q^2 \Rightarrow q^2 = 3k^2$

This shows that $q^2$ is also a multiple of $\Rightarrow$

$q$ is a multiple of 3.

We have shown that both $p$ and $q$ are divisible by 3

then: $PGCD(p, q) \neq 1$

Our assumption is false, so $\sqrt{3}$ cannot be rational

Therefore, $\sqrt{3}$ is irrational.

4) by induction proof: $\forall n \in \mathbb{N}^* : 1 + 2^3 + 3^3 + \dots + n^3 = \dfrac{n^2(n+1)^2}{4}$

* for $n = 1$, we have: $1 = \dfrac{1^2(1+1)^2}{4} \Rightarrow 1 = 1$ (True)

* let $n \geqslant 1$, we assume that:

$$1 + 2^3 + \dots + n^2 = \frac{n^2(n+1)^2}{4} \text{ is true}$$

and we show that:

$$1 + 2^3 + \dots + (n+1)^3 = \frac{(n+1)^2(n+2)^2}{4}$$

let $1 + 2^3 + \dots + n^2 + (n+1)^3 = \dfrac{n^2(n+1)^2}{4} + (n+1)^3$

$$= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} = \frac{(n+1)^2(n^2+4n+4)}{4}$$

$$= \frac{(n+1)^2(n+2)^2}{4}$$

Then: $\forall n \geqslant 1 : 1 + 2^3 + \dots + n^3 = \dfrac{n^2(n+1)^2}{4}$.