This chapter is an introduction to analog filters. It begins with the basic analog filters, transfer functions, and frequency response. The amplitude characteristics of Butterworth and Chebychev filters. It concludes with design examples using MATLAB.

11.1 Filter Types and Classifications

Analog filters are defined over a continuous range of frequencies. They are classified as ***low-pass, high-pass, band-pass*** and ***band-elimination (stop-band).*** The ideal amplitude characteristics of each are shown in Figure 11.1. The ideal characteristics are not physically realizable; we will see that practical filters can be designed to approximate these characteristics.

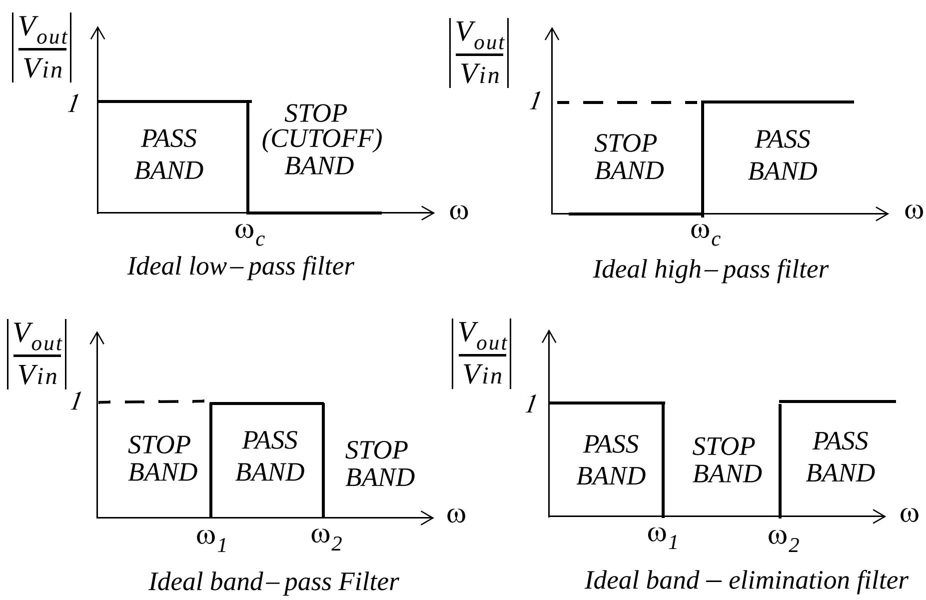

Figure 11.1. Amplitude characteristics of the types of filters

Another, less frequently mentioned filter, is the *all-pass* or *phase shift* filter. It has a constant amplitude response but is phase varies with frequency. Please refer to Exercise 4.

A *digital filter*, in general, is a computational process, or algorithm that converts one sequence of numbers representing the input signal into another sequence representing the output signal. Accordingly, a digital filter can perform functions as differentiation, integration, estimation, and, of course, like an analog filter, it can filter out unwanted bands of frequency.

Analog filter functions have been used extensively as prototype models for designing digital filters and, therefore, we will present them first.

11.2 Basic Analog Filters

An analog filter can also be classified as *passive* or *active*. Passive filters consist of passive devices such as resistors, capacitors and inductors. Active filters are, generally, operational amplifiers with resistors and capacitors connected to them externally. We can find out whether a filter, passive or active, is a low-pass, high-pass, etc., from its the frequency response that can be obtained from its transfer function. The procedure is illustrated with the examples that follow.

Example 11.1

Derive expressions for the magnitude and phase responses of the series RC network of Figure 11.2, and sketch their characteristics.

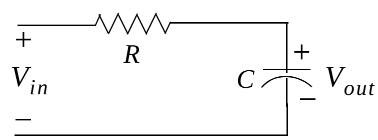

Figure 11.2. Series RC network for Example 11.1

Solution:

$$V_{out} = \frac{1/j\omega C}{R + 1/j\omega C} V_{in}$$

or

$$G(j\omega) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + j\omega RC} = \frac{1}{(\sqrt{1 + \omega^2 R^2 C^2}) \angle atan(\omega RC)}$$
$$= \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \angle -atan(\omega RC) \qquad (11.1)$$

The magnitude of (11.1) is

$$\boxed{|G(j\omega)| = \left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}} \qquad (11.2)$$

and the phase angle, also known as the *argument*, is

$$\theta = arg\{G(j\omega)\} = arg\left(\frac{V_{out}}{V_{in}}\right) = -atan(\omega RC) \tag{11.3}$$

We can obtain a quick sketch for the magnitude $|G(j\omega)|$ versus ω by evaluating (11.2) at $\omega = 0$, $\omega = 1/RC$, and $\omega \to \infty$. Thus,

as $\omega \to 0$,

$$|G(j\omega)| \cong 1$$

for $\omega = 1/RC$,

$$|G(j\omega)| = 1/\sqrt{2} = 0.707$$

and as $\omega \to \infty$,

$$|G(j\omega)| \cong 0$$

We will use the MATLAB code below to plot $|G(j\omega)|$ versus radian frequency ω. This is shown in Figure 11.3 where, for convenience, we let $RC = 1$.

```
w=0:0.02:10; RC=1; magGs=1./sqrt(1+w.*RC); semilogx(w,magGs); grid
```

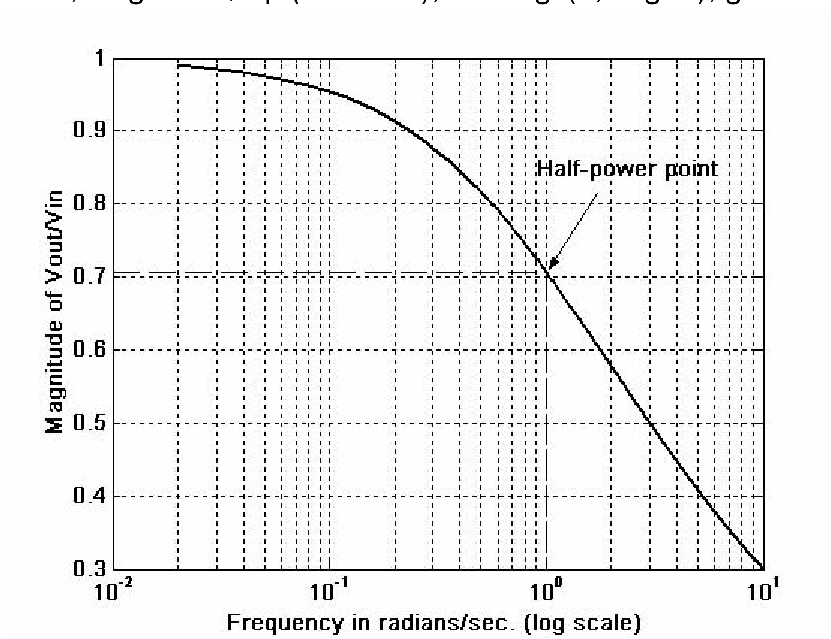

Figure 11.3. Amplitude characteristics of a series RC low-pass filter with $RC = 1$

We can also obtain a quick sketch for the phase angle, i.e., $\theta = arg\{G(j\omega)\}$ versus ω by evaluating of (11.3) at $\omega = 0$, $\omega = 1/RC$, $\omega = -1/RC$, $\omega \to -\infty$ and $\omega \to \infty$. Thus,

As $\omega \to 0$,

$$\theta \cong -atan 0 \cong 0°$$

For $\omega = 1/RC$,

$$\theta = -atan1 = -45°$$

For $\omega = -1/RC$,

$$\theta = -atan(-1) = 45°$$

As $\omega \rightarrow -\infty$,

$$\theta = -atan(-\infty) = 90°$$

and as $\omega \rightarrow \infty$,

$$\theta = -atan(\infty) = -90°$$

We will use the MATLAB code below to plot the phase angle θ versus radian frequency ω. This is shown in Figure 11.3 where, for convenience, we let $RC = 1$.

```
w=-8:0.02:8; RC=1; argGs=–atan(w.*RC).*180./pi; plot(w,argGs); grid
```

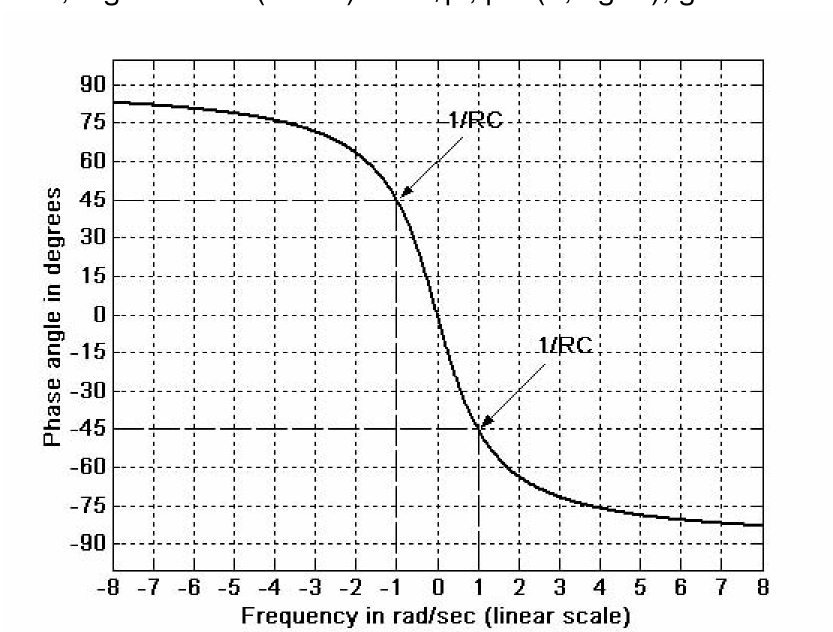

Figure 11.4. Phase characteristics of a series RC low-pass filter with $RC = 1$

Example 11.2

The network of Figure 11.5 is also a series RC circuit, where the positions of the resistor and capacitor have been interchanged. Derive expressions for the magnitude and phase responses, and sketch their characteristics.

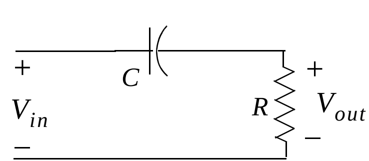

Figure 11.5. RC network for Example 11.2

Solution:

$$V_{out} = \frac{R}{R+1/j\omega C} V_{in}$$

or

$$G(j\omega) = \frac{V_{out}}{V_{in}} = \frac{j\omega RC}{1+j\omega RC} = \frac{j\omega RC + \omega^2 R^2 C^2}{1+\omega^2 R^2 C^2} = \frac{\omega RC(j+\omega RC)}{1+\omega^2 R^2 C^2}$$
$$= \frac{\omega RC\sqrt{1+\omega^2 R^2 C^2}\angle atan(1/(\omega RC))}{1+\omega^2 R^2 C^2} = \frac{1}{\sqrt{1+1/(\omega^2 R^2 C^2)}}\angle atan(1/(\omega RC)) \quad (11.4)$$

The magnitude of (11.4) is

$$\boxed{|G(j\omega)| = \frac{1}{\sqrt{1+1/(\omega^2 R^2 C^2)}}} \quad (11.5)$$

and the phase angle or argument, is

$$\boxed{\theta = arg\{G(j\omega)\} = atan(1/\omega RC)} \quad (11.6)$$

We can obtain a quick sketch for the magnitude $|G(j\omega)|$ versus ω by evaluating (11.5) at $\omega = 0$, $\omega = 1/RC$, and $\omega \to \infty$. Thus,

As $\omega \to 0$,

$$|G(j\omega)| \cong 0$$

For $\omega = 1/RC$,

$$|G(j\omega)| = 1/\sqrt{2} = 0.707$$

and as $\omega \to \infty$,

$$|G(j\omega)| \cong 1$$

We will use the MATLAB code below to plot $|G(j\omega)|$ versus radian frequency ω. This is shown in Figure 11.5 where, for convenience, we let $RC = 1$.

```
w=0:0.02:100; RC=1; magGs=1./sqrt(1+1./(w.*RC).^2); semilogx(w,magGs); grid
```

We can also obtain a quick sketch for the phase angle, i.e., $\theta = arg\{G(j\omega)\}$ versus ω, by evaluating (11.6) at $\omega = 0$, $\omega = 1/RC$, $\omega = -1/RC$, $\omega \to -\infty$, and $\omega \to \infty$. Thus,

As $\omega \to 0$,

$$\theta \cong -atan 0 \cong 0°$$

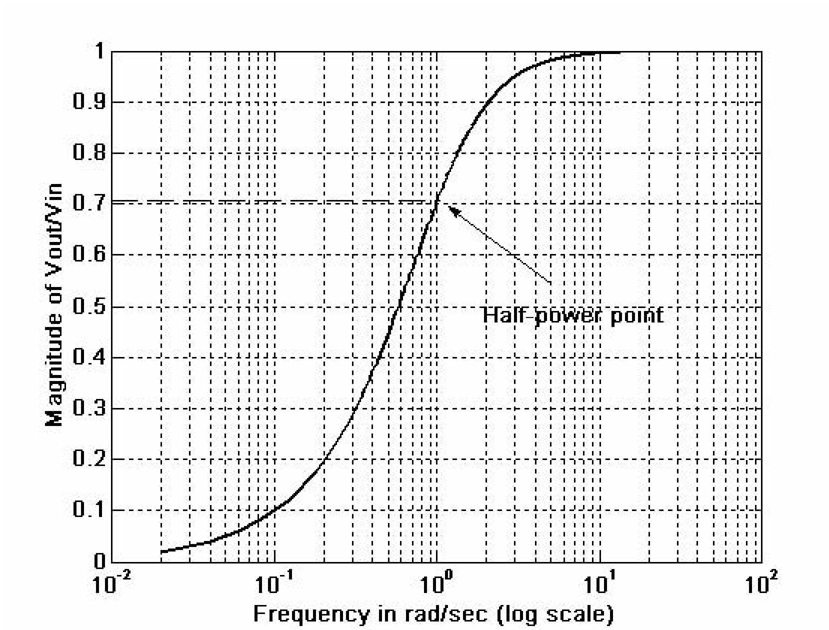

Figure 11.6. Amplitude characteristics of a series RC high-pass filter with $RC = 1$

For $\omega = 1/RC$,

$$\theta = -atan1 = -45°$$

For $\omega = -1/RC$,

$$\theta = -atan(-1) = 45°$$

As $\omega \to -\infty$,

$$\theta = -atan(-\infty) = 90°$$

and as $\omega \to \infty$,

$$\theta = -atan(\infty) = -90°$$

We will use the MATLAB code below to plot the phase angle θ versus radian frequency ω. This is shown in Figure 11.7 where, for convenience, we let $RC = 1$.

```
w=-8:0.02:8; RC=1; argGs=atan(1./(w.*RC)).*180./pi; plot(w,argGs); grid
```

Other low-pass, high-pass, band-pass, and band-elimination passive filters consist of combinations of resistors, inductors, and capacitors. They are given as exercises in Chapter 4.

Example 11.3

The circuit of Figure 11.8 is an active low-pass filter and its magnitude $|G(j\omega)|$ versus ω was derived and plotted in Example 4.7 of Chapter 4.

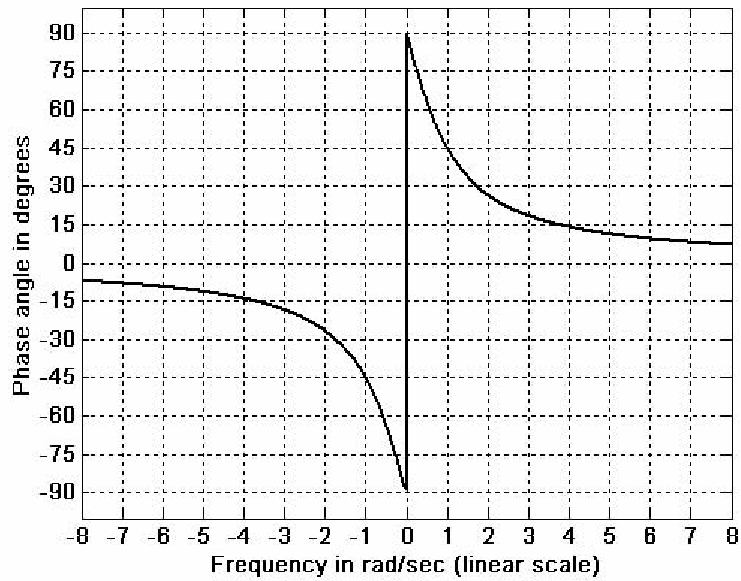

Figure 11.7. Phase characteristics of an RC high-pass filter with $RC = 1$

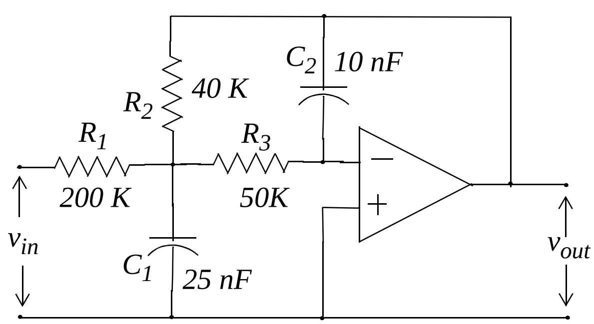

Figure 11.8. Low-pass filter for Example 11.3

Using the derivation procedures of the examples we discussed thus far, we can analyze any circuit to determine its behavior. However, as we found out in Example 4.7, the derivations are tedious. Fortunately, several books containing filters and their characteristics, have been published. We will discuss these later.

In our subsequent discussion we will be concerned with filter design.

11.3 Low-Pass Analog Filters

In this section, we will use the analog low-pass filter as a basis. We will see later that, using transformations, we can derive high-pass and the other types of filters from a basic low-pass filter. We will discuss the *Butterworth, Chebyshev* and *Cauer* (*elliptic*) filters.

The first step in the design of an analog low-pass filter is to find a suitable *amplitude-squared function* $A^2(\omega)$, and from it derive a $G(s)$ function such that

$$A^2(\omega) = G(s) \cdot G(-s)|_{s = j\omega} \tag{11.7}$$

Since $(j\omega)^* = (-j\omega)$, the square of the magnitude of a complex number can be expressed as that number and its complex conjugate. Thus, if the magnitude is A, then

$$A^2(\omega) = |G(j\omega)| = G(j\omega)G^*(j\omega) = G(j\omega) \cdot G(-j\omega) \tag{11.8}$$

Now, $G(j\omega)$ can be considered as $G(s)$ evaluated at $s = j\omega$, and thus (11.7) is justified. Also, since A is understood to represent the magnitude, it needs not be enclosed in vertical lines.

Not all amplitude-squared functions can be decomposed to $G(s)$ and $G(-s)$ rational functions; only even functions of ω, positive for all ω, and *proper rational functions*[*] can satisfy (11.7).

Example 11.4

It is given that

$$G(s) = \frac{3s^2 + 5s + 7}{s^2 + 4s + 6}$$

Compute $A^2(\omega)$.

Solution:

Since

$$G(s) = \frac{3s^2 + 5s + 7}{s^2 + 4s + 6}$$

it follows that

$$G(-s) = \frac{3s^2 - 5s + 7}{s^2 - 4s + 6}$$

and

$$G(s) \cdot G(-s) = \frac{3s^2 + 5s + 7}{s^2 + 4s + 6} \cdot \frac{3s^2 - 5s + 7}{s^2 - 4s + 6} = \frac{9s^4 + 17s^2 + 49}{s^4 - 4s^2 + 36}$$

Therefore,

* *It was stated earlier, that a rational function is said to be proper if the largest power in the denominator is equal to or larger than that of the numerator.*

$$A^2(\omega) = G(s)\cdot G(-s)\big|_{s=j\omega} = \left.\frac{9s^4 + 17s^2 + 49}{s^4 - 4s^2 + 36}\right|_{s=j\omega} = \frac{9\omega^4 - 17\omega^2 + 49}{\omega^4 + 4\omega^2 + 36}$$

The general form of the amplitude square function $A^2(\omega)$ is

$$A^2(\omega) = \frac{C(b_k\omega^{2k} + b_{k-1}\omega^{2k-2} + \ldots + b_0)}{a_k\omega^{2k} + a_{k-1}\omega^{2k-2} + \ldots + a_0} \tag{11.9}$$

where C is the DC gain, a and b are constant coefficients, and k is a positive integer denoting the order of the filter. Once the amplitude square function $A^2(\omega)$ is known, we can find $G(s)$ from (11.9) with the substitution $(j\omega)^2 = -\omega^2 = s^2$ or $\omega^2 = -s^2$, that is,

$$G(s)\cdot G(-s) = A^2(\omega)\big|_{\omega^2 = -s^2} \tag{11.10}$$

In the simplest low-pass filter, the DC gain of the amplitude square function is unity. In this case (11.9) reduces to

$$A^2(\omega) = \frac{b_0}{a_k\omega^{2k} + a_{k-1}\omega^{2k-2} + \ldots + a_0} \tag{11.11}$$

and at high frequencies reduces to

$$A^2(\omega) \approx \frac{b_0/a_k}{\omega^{2k}} \tag{11.12}$$

The *attenuation rate* of this approximation is $6k\ dB/octave$ or $20k\ dB/decade$. To understand this, let us review the definitions of *octave* and *decade.*

Consider two frequencies u_1 and u_2. Let

$$u_2 - u_1 = log_{10}\omega_2 - log_{10}\omega_1 = log_{10}\frac{\omega_2}{\omega_1} \tag{11.13}$$

If these frequencies are such that $\omega_2 = 2\omega_1$, we say that these frequencies are separated by one octave, and if $\omega_2 = 10\omega_1$, they are separated by one decade.

To compute the attenuation rate of (11.12), we take the square root of both sides. Then,

$$A(\omega) = \frac{\sqrt{b_0/a_k}}{\omega^k} = \frac{Constant}{\omega^k} = \frac{B}{\omega^k} \tag{11.14}$$

Taking the common log of both sides of (11.14) and multiplying by 20, we get

$$20log_{10}A(\omega) = 20log_{10}B - 20log_{10}\omega^k = -20klog_{10}\omega + 20log_{10}B \qquad (11.15)$$

or

$$A(\omega)_{dB} = -20klog_{10}\omega + 20log_{10}B \qquad (11.16)$$

This is an equation of a straight line with $slope = -20k \ \ dB/decade$, and $intercept = B$ as shown in Figure 11.9.

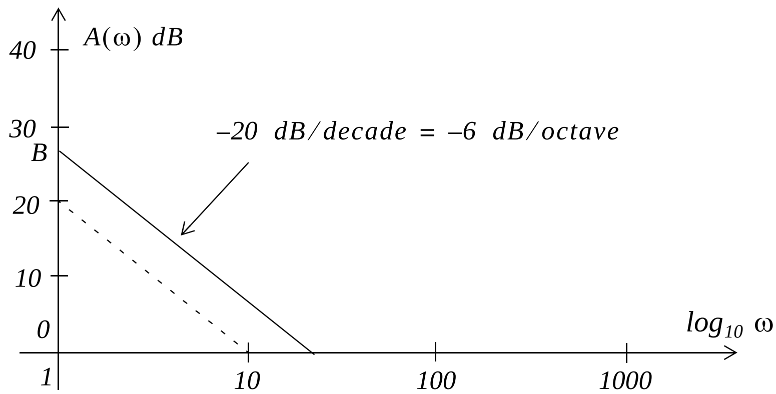

Figure 11.9. The $-20 \ \ dB/decade = -6 \ \ dB/octave$ line

The procedure of finding the transfer function $G(s)$ from the amplitude square function $A^2(\omega)$, is best illustrated with the following example.

Example 11.5

Given the amplitude square function

$$A^2(\omega) = \frac{16(-\omega^2+1)}{(\omega^2+4)(\omega^2+9)} \qquad (11.17)$$

find a suitable transfer function $G(s)$

Solution:

From (11.10),

$$G(s)G(-s) = A^2(\omega)\Big|_{\omega^2 = -s^2} = \frac{16(s^2+1)}{(-s^2+4)(-s^2+9)} \qquad (11.18)$$

This function has zeros at $s = \pm j1$, and poles at $s = \pm 2$ and $s = \pm 3$.

There is no restriction on the zeros but, for stability*, we select the left-half s-plane poles.

We must also select the gain constant such that $G(0) = A(0)$.

Let

$$G(s) = \frac{K(s^2+1)}{(s+2)(s+3)} \tag{11.19}$$

We must find K such that $G(0) = A(0)$. From (11.17),

$$A^2(0) = 16/36 = 4/9$$

or

$$A(0) = 2/3$$

From (11.19),

$$G(0) = K/6$$

and for $G(0) = A(0)$ we must have,

$$K/6 = 2/3$$

or

$$K = 12/3 = 4$$

By substitution into (11.19),

$$G(s) = 4\frac{(s^2+1)}{(s+2)(s+3)}$$

11.4 Design of Butterworth Analog Low-Pass Filters

We will consider the *Butterworth low-pass filter* whose amplitude-squared function is

$$A^2(\omega) = \frac{1}{(\omega/\omega_C)^{2k}+1} \tag{11.20}$$

where k is a positive integer, and ω_C is the cutoff ($3\ dB$) frequency. Figure 11.10 shows relation (11.20) for $k = 1, 2, 4,$ and 8. The plot was created with the following MATLAB code.

```
w_w0=0:0.02:3; Aw2k1=sqrt(1./(w_w0.^2+1)); Aw2k2=sqrt(1./(w_w0.^4+1));...
Aw2k4=sqrt(1./(w_w0.^8+1)); Aw2k8=sqrt(1./(w_w0.^16+1));...
plot(w_w0,Aw2k1,w_w0,Aw2k2,w_w0,Aw2k4,w_w0,Aw2k8); grid
```

* *Generally, a system is said to be stable if a finite input produces a finite output. Alternately, a system is stable if the impulse response $h(t)$ vanishes after a sufficiently long time. Stability is discussed in Feedback and Control Systems textbooks.*

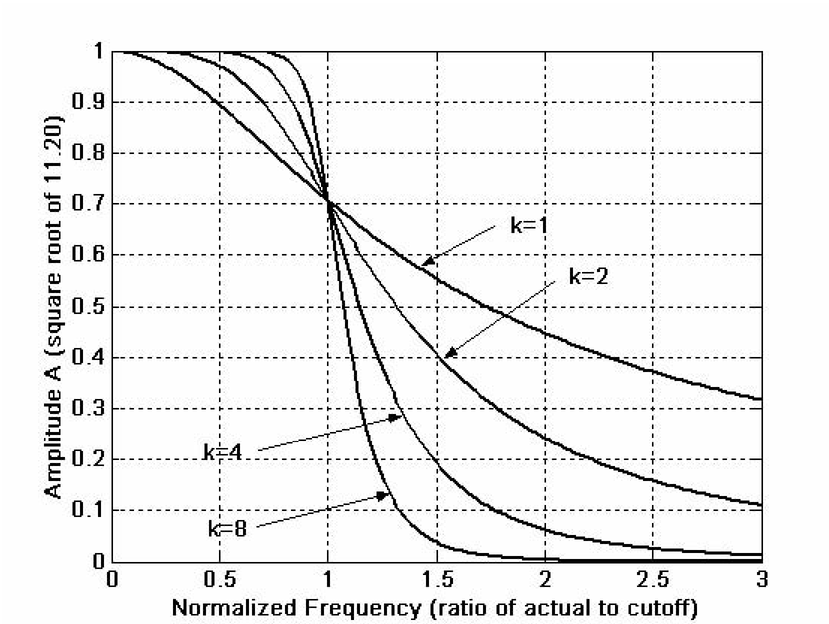

Figure 11.10. Butterworth low-pass filter amplitude characteristics

All Butterworth filters have the property that all poles of the transfer functions that describes them, lie on a circumference of a circle of radius ω_C, and they are $2\pi/2k$ radians apart. Thus, if $k = odd$, the poles start at zero radians, and if $k = even$, they start at $2\pi/2k$. But regardless whether k is odd or even, the poles are distributed in symmetry with respect to the $j\omega$ axis. For stability, we choose the left half-plane poles to form $G(s)$.

We can find the nth roots of a the complex number s by *DeMoivre's theorem*. It states that

$$\sqrt[n]{re^{j\theta}} = \sqrt[n]{r}e^{j\left(\frac{\theta + 2k\pi}{n}\right)} \qquad k = 0, \pm 1, \pm 2, \ldots \tag{11.21}$$

Example 11.6

Derive the transfer function $G(s)$ for the third order ($k = 3$) Butterworth low-pass filter with *normalized cutoff frequency* $\omega_C = 1\ rad/s$.

Solution:

With $k = 3$ and $\omega_c = 1\ rad/s$, (11.20) reduces to

$$A^2(\omega) = \frac{1}{\omega^6 + 1} \tag{11.22}$$

With the substitution $\omega^2 = -s^2$, (11.22) becomes

$$G(s) \cdot G(-s) = \frac{1}{-s^6 + 1} \tag{11.23}$$

Then, $s = \sqrt[6]{1} \angle 0°$ and by DeMoivre's theorem, with $n = 6$,

$$\sqrt[6]{1e^{j0}} = \sqrt[6]{1}e^{j\left(\frac{0 + 2k\pi}{6}\right)} \quad k = 0, 1, 2, 3, 4, 5$$

Thus,

$$s_1 = 1\angle 0° = 1$$

$$s_2 = 1\angle 60° = \frac{1}{2} + j\frac{\sqrt{3}}{2}$$

$$s_3 = 1\angle 120° = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$$

$$s_4 = 1\angle 180° = -1$$

$$s_5 = 1\angle 240° = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$$

$$s_6 = 1\angle 300° = \frac{1}{2} - j\frac{\sqrt{3}}{2}$$

As expected, these six poles lie on the circumference of the circle with radius $\omega_c = 1$ as shown in Figure 11.11.

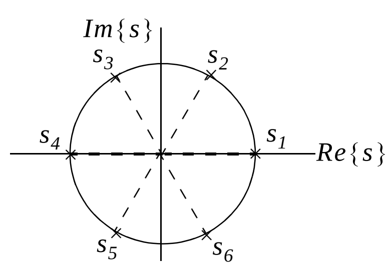

Figure 11.11. Location of the poles for the transfer function of Example 11.6

The transfer function $G(s)$ is formed with the left half-plane poles s_3, s_4, and s_5. Then,

$$G(s) = \frac{K}{\left(s + \frac{1}{2} - j\frac{\sqrt{3}}{2}\right)(s + 1)\left(s + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right)} \tag{11.24}$$

We use MATLAB to express the denominator as a polynomial.

```
syms s; den=(s+1/2–sqrt(3)*j/2)*(s+1)*(s+1/2+sqrt(3)*j/2)
```

```
den =
 (s+1/2-1/2*i*3^(1/2))*(s+1)*(s+1/2+1/2*i*3^(1/2))
```

```
expand(den)
```

```
ans =
s^3+2*s^2+2*s+1
```

Therefore, (11.24) simplifies to

$$G(s) = \frac{K}{s^3 + 2s^2 + 2s + 1} \tag{11.25}$$

The gain K is found from $A^2(0) = 1$ or $A(0) = 1$ and $G(0) = K$. Thus, $K = 1$ and

$$G(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} \tag{11.26}$$

and this is the transfer function $G(s)$ for the third order ($k = 3$) Butterworth low-pass filter with normalized cutoff frequency $\omega_C = 1\ rad/s$.

The general form of any analog low-pass (Butterworth, Chebyshev, Elliptic, etc.) filter is

$$\boxed{G(s)|_{lp} = \frac{b_0}{a_k s^k + \ldots + a_2 s^2 + a_1 s + a_0}} \tag{11.27}$$

The pole locations and the coefficients of the corresponding denominator polynomials, have been derived and tabulated by Weinberg in *Network Analysis and Synthesis,* McGraw-Hill.

Table 11.1 shows the first through the fifth order coefficients for Butterworth analog low-pass filter denominator polynomials with normalized frequency $\omega_C = 1\ rad/s$.

TABLE 11.1 Values for the coefficients a_i in (11.27)

Coefficients of Denominator Polynomial for Butterworth Low-Pass Filters						
Order	a_5	a_4	a_3	a_2	a_1	a_0
1						1
2				1	1.4142136	1
3			1	2	2	1
4		2.6131259	3.1442136	2.6131259	1	1
5	1	3.2360680	5.2360680	5.2360680	3.2360680	1

We can also use the MATLAB **buttap** and **zp2tf** functions. The first returns the zeros, poles, and gain for an $N-th$ order normalized prototype Butterworth analog low-pass filter. The resulting filter has N poles around the unit circle in the left half plane, and no zeros. The second performs the zero-pole to transfer function conversion.

Example 11.7

Use MATLAB to find the numerator b and denominator a coefficients for the third-order Butterworth low-pass filter prototype with normalized cutoff frequency*.

Solution:

```
[z,p,k]=buttap(3); [b,a]=zp2tf(z,p,k)
```

```
b =
     0     0     0     1
a =
    1.0000    2.0000    2.0000    1.0000
```

We observe that the denominator coefficients are the same as in Table 11.1.

Table 11.2 shows factored forms of the denominator polynomials in terms of linear and quadratic factors with normalized frequency $\omega_C = 1\ rad/s$.

TABLE 11.2 Factored forms for Butterworth low-pass filters

Denominator in Factored form for Butterworth Low-Pass Filters with $\omega_C = 1\ rad/s$	
k	Denominator of Equation (11.27)
1	$s+1$
2	$s^2+1.4142s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.7654s+1)(s^2+1.8478s+1)$
5	$(s+1)(s^2+0.6180s+1)(s^2+1.6180s+1)$
6	$(s^2+05176s+1)(s^2+1.4142s+1)(s^2+1.9318s+1)$
7	$(s+1)(s^2+0.4449s+1)(s^2+1.2465s+1)(s^2+1.8022s+1)$
8	$(s^2+0.3896s+1)(s^2+1.1110s+1)(s^2+1.6630s+1)(s^2+1.9622s+1)$

* *Henceforth, normalized cutoff frequency will be understood to be* $\omega_C = 1\ rad/s$

The equations shown in Table 11.2 can be derived from

$$G(s) = \frac{1}{(-1)^n \prod_{i=0}^{n-1} \left(\frac{s}{s_i} - 1\right)} \tag{11.28}$$

where the factor $(-1)^n$ is to ensure that $G(0) = 1$, and s_i denotes the poles on the left half of the s-plane. They can be found from

$$s_i = \omega_C\left(-sin\frac{(2i+1)\pi}{2k} + j cos\frac{(2i+1)\pi}{2k}\right) \tag{11.29}$$

We must remember that the factors in Table 11.2 apply only when the cutoff frequency is normalized to $\omega_C = 1\ rad/s$. If $\omega_C \neq 1$, we must scale the transfer function appropriately.

We can convert to the actual transfer function using the relation

$$G(s)_{actual} = G\left(\frac{\omega_{norm} \times s}{\omega_{actual}}\right)$$

and since, usually $\omega_{norm} = 1\ \ rad/s$,

$$G(s)_{actual} = G\left(\frac{s}{\omega_{actual}}\right) \tag{11.30}$$

that is, we replace s with s/ω_{actual}

Quite often, we require that $\omega \geq \omega_C$, that is, in the stop band of the low-pass filter, the attenuation to be larger than $-20\ dB/decade$, i.e., we require a sharper cutoff. As we have seen from the plots of Figure 11.10, the Butterworth low-pass filter cutoff becomes sharper for larger values of k. Accordingly, we generate the plot for different values of k shown in Figure 11.12 using the following MATLAB code.

```
w_w0=1:0.02:10; dBk1=20.*log10(sqrt(1./(w_w0.^2+1)));...
dBk2=20.*log10(sqrt(1./(w_w0.^4+1))); dBk3=20.*log10(sqrt(1./(w_w0.^6+1)));...
dBk4=20.*log10(sqrt(1./(w_w0.^8+1))); dBk5=20.*log10(sqrt(1./(w_w0.^10+1)));...
dBk6=20.*log10(sqrt(1./(w_w0.^12+1))); dBk7=20.*log10(sqrt(1./(w_w0.^14+1)));...
dBk8=20.*log10(sqrt(1./(w_w0.^16+1))); semilogx(w_w0,dBk1,w_w0,dBk2,w_w0,dBk3,...
w_w0,dBk4,w_w0,dBk5,w_w0,dBk6,w_w0,dBk7,w_w0,dBk8); grid
```

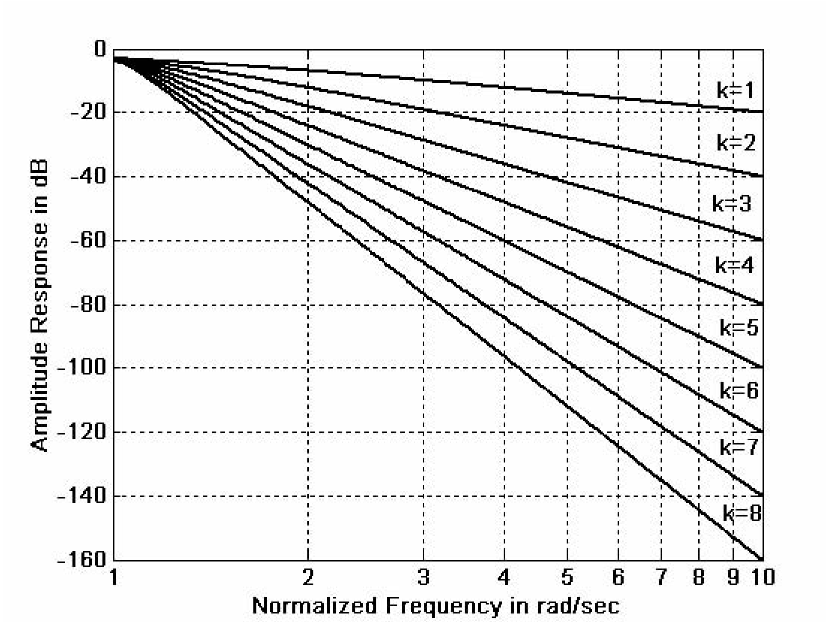

Figure 11.12. Attenuation for different values of k

Example 11.8

Using the attenuation curves of Figure 11.12, derive the transfer function of a Butterworth low-pass analog filter with pass band bandwidth of $5\ rad/s$, and attenuation in the stop band at least $30\ dB/decade$ for frequencies larger than $15\ rad/s$.

Solution:

We refer to Figure 11.12 and at $\omega/\omega_C = 15/5 = 3$, we see that the vertical line at this value crosses the $k = 3$ curve at approximately $-28\ dB$, and the $k = 4$ curve at approximately $-37\ dB$. Since we require that the attenuation be at least $-30\ dB$, we use the attenuation corresponding to the $k = 4$ curve. Accordingly, we choose a fourth order Butterworth low-pass filter whose normalized transfer function, from Table 11.2, is

$$G(s)_{norm} = \frac{1}{(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)} \tag{11.31}$$

and since $\omega_C = 5\ \ rad/s$, we replace s with $s/5$. Then,

$$G(s)_{actual} = \frac{1}{\left(\frac{s^2}{25} + \frac{0.7654s}{5} + 1\right)\left(\frac{s^2}{25} + \frac{1.8478s}{5} + 1\right)}$$

or

$$G(s)_{actual} = \frac{625}{(s^2 + 3.8270s + 25)(s^2 + 9.2390s + 25)} = \frac{625}{s^4 + 13.066s^3 + 85.358s^2 + 326.650s + 625} \quad (11.32)$$

As a last step, we wish to know how to design a circuit (passive or active), that will satisfy a transfer function such as the one above. Fortunately, the work for us has been done by others who have developed analog filter prototypes, both passive and active.

Some good references are:
Electronic Filter Design Handbook, Williams and Taylor, McGraw-Hill
Electronic Engineers' Handbook, Fink and Christiansen, McGraw-Hill
Reference Data for Engineers Handbook, Van Valkenburgh, Howard Sams

As an example, the *Reference Data for Engineers Handbook* provides the circuit of Figure 11.13 that is known as *Second Order Voltage Controlled Voltage Source (VCVS)* low-pass filter.

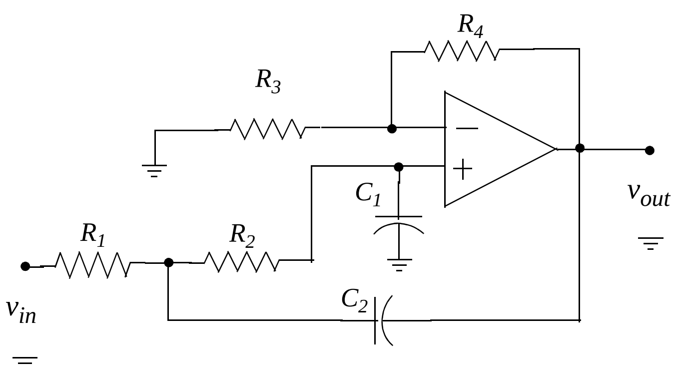

Figure 11.13. VCVS low-pass filter

The transfer function of the second order VCVS low-pass filter of Figure 11.13 is given as

$$G(s) = \frac{Kb\omega_c^2}{s^2 + a\omega_c s + b\omega_c^2} \quad (11.33)$$

This is referred to as a second order ***all-pole***[*] approximation to the ideal low-pass filter with cutoff frequency ω_C, where K is the gain, and the coefficients a and b are provided by tables.

For a non-inverting positive gain K, the circuit of Figure 11.13 satisfies the transfer function of (11.33) with the conditions that

* *The terminology "all-pole" stems from the fact that the s–plane contains poles only and the zeros are at $\pm\infty$, that is, the s–plane is all poles and no zeros.*

$$R_1 = \frac{2}{\left\{aC_2 + (\sqrt{[a^2 + 4b(K-1)]C_2^2 - 4bC_1C_2})\right\}\omega_c} \tag{11.34}$$

$$R_2 = \frac{1}{bC_1C_2R_1\omega_c^2} \tag{11.35}$$

$$R_3 = \frac{K(R_1 + R_2)}{(K-1)} \qquad K \neq 1 \tag{11.36}$$

$$R_4 = K(R_1 + R_2) \tag{11.37}$$

From (11.36) and (11.37), we observe that $K = 1 + R_4/R_3$.

A fourth-order all-pole low-pass filter transfer function is a ratio of a constant to a fourth degree polynomial. A practical method of obtaining a fourth order transfer function, is to factor it into two second-order transfer functions of the form of (11.33), i.e.,

$$G(s) = \frac{K_1 b_1 \omega_c^2}{s^2 + a_1\omega_c s + b_1\omega_c^2} \cdot \frac{K_2 b_2 \omega_c^2}{s^2 + a_2\omega_c s + b_2\omega_c^2} \tag{11.38}$$

Each factor in (11.38) can be realized by a stage (circuit). Then, the two stages can be cascaded as shown in Figure 11.14.

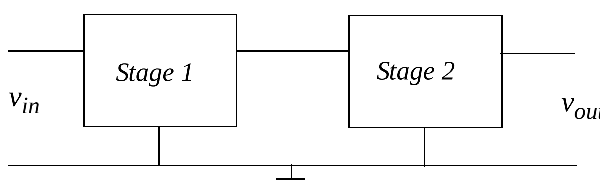

Figure 11.14. Cascaded stages

Table 11.3 lists the Butterworth low-pass coefficients for second and fourth-order designs, where a and b apply to the transfer functions of (11.33) and (11.38).

For a practical design of a second-order VCVS circuit, we select standard values for capacitors C_1 and C_2 of the circuit of Figure 11.13, we substitute the appropriate values for the coefficients a and b from Table 11.3, we choose desired values for the gain K and cutoff frequency ω_C, and we substitute these in (11.34) through (11.37) to find the values of the resistors R_1 through R_4.

TABLE 11.3 Coefficients for Butterworth low-pass filter designs

Coefficients for Second and Fourth Order Butterworth Low-Pass Filter Designs		
Order		
2	a	1.41421
	b	1.0000
4	a_1	0.76537
	b_1	1.0000
	a_2	1.84776
	b_2	1.0000

Example 11.9

Design a second-order VCVS Butterworth low-pass filter with gain $K = 2$ and cutoff frequency $f_C = 1\ kHz$.

Solution:

We will use the second order VCVS prototype op amp circuit of Figure 11.13, with capacitance values $C_1 = C_2 = 0.01\ \mu F = 10^{-8}\ F$. From Table 11.3, $a = 1.41421 = \sqrt{2}$ and $b = 1$.

We substitute these values into (11.34) through (11.37), to find the values of the resistors.

We use MATLAB to do the calculations as follows:

```
C1=10^(-8); C2=C1; a=sqrt(2); b=1; K=2; wc=2*pi*10^3;
% and from (11.34) through (11.37)
R1=2/((a*C2+sqrt((a^2+4*b*(K–1))*C2^2-4*b*C1*C2))*wc);
R2=1/(b*C1*C2*R1*wc^2);  R3=K*(R1+R2)/(K-1); R4=K*(R1+R2); fprintf(' \n');...
fprintf('R1 = %6.0f \t',R1); fprintf('R2 = %6.0f \t',R2);...
fprintf('R3 = %6.0f \t',R3); fprintf('R4 = %6.0f \t',R4)
```

```
R1 = 11254  R2 = 22508  R3 = 67524  R4 = 67524
```

These are the calculated values but they are not standard resistor values; we must select standard resistor values as close as possible to the calculated values.

It will be interesting to find out what the frequency response of this filter looks like, with capacitors $C_1 = C_2 = 0.01\ \mu F$ and standard 1% tolerance resistors with values $R_1 = 11.3\ K\Omega$, $R_2 = 2 \times R_1 = 22.6\ K\Omega$, and $R_3 = R_4 = 68.1\ K\Omega$.

We now substitute these values into the equations of (11.34) through (11.37), and we solve the first equation of this set for the cutoff frequency ω_C. Then, we use ω_C with the transfer function of (11.33). We do this with the following MATLAB code which produces the plot of Figure 11.15.

```
f=1:10:5000; R1=11300; R2=22600; R3=68100; R4=R3; C1=10^(−8); C2=C1;
a=sqrt(2); b=1; w=2*pi*f; fc=sqrt(1/(b*R1*R2*C1*C2))/(2*pi); wc=2*pi*fc;
K=1+R3/R4; s=w*j; Gw=(K.*b.*wc.^2)./(s.^2+a.*wc.*s+b.*wc.^2); magGw=abs(Gw);
semilogx(f,magGw); grid; hold on; xlabel('Frequency Hz'); ylabel('|Vout/Vin|');
title ('2nd Order Butterworth Low-Pass Filter Response')
```

The frequency response of this low-pass filter is shown in Figure 11.15.

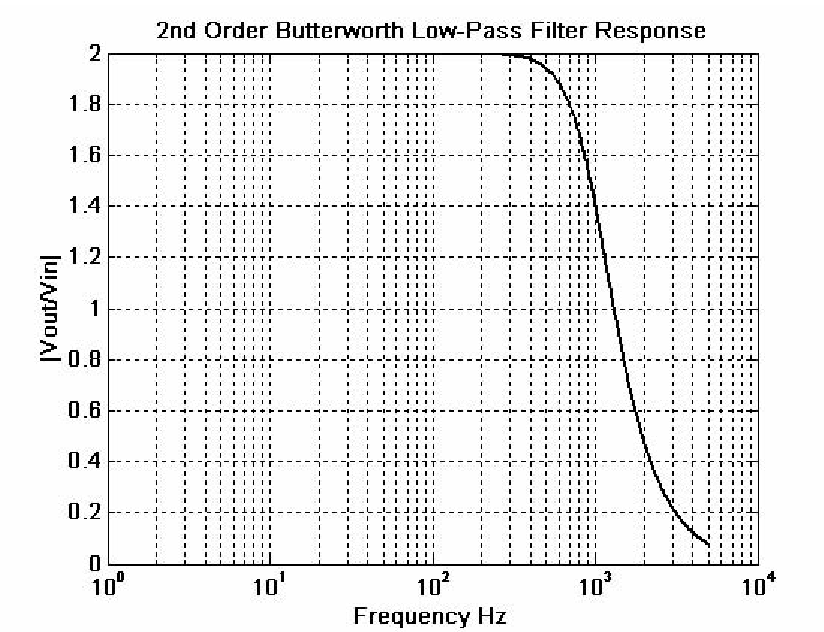

Figure 11.15. Plot for the VCVS low-pass filter of Example 11.9

We see that the cutoff frequency occurs at about $1\ KHz$.

We have used the MATLAB **buttap** function earlier to aid us in the design of Butterworth filters with the cutoff frequency normalized to $1\ rad/s$. We can also use the **bode** function to display both the (asymptotic) magnitude and phase plots. The following code will produce the Bode magnitude and phase plots for a two-pole Butterworth low-pass filter.

```
[z,p,k]= buttap(2);                         % Specify a two-pole filter
[b,a]=zp2tf(z,p,k);                         % Display in polynomial rational form
w=0:0.01:4; [mag,phase]=bode(b,a,w);
b                                           % Display b coefficients
```

```
b =
     0     0     1
```

```
a                                          % Display a coefficients

a =
    1.0000    1.4142    1.0000

num=[0  0  1]; den=[1  sqrt(2)  1];
bode(num,den); title('Butterworth 2nd Order Low-Pass Filter')
```

The Bode plots are shown in Figure 11.16.

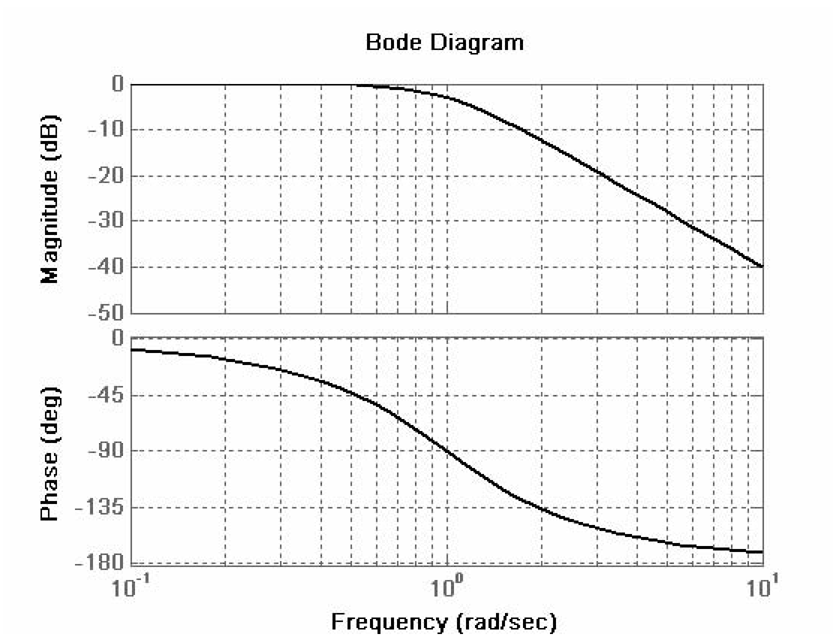

Figure 11.16. Bode plots for example 11.9

11.5 Design of Type I Chebyshev Analog Low-Pass Filters

The Type I Chebyshev filters are based on approximations derived from the Chebyshev polynomials $C_k(x)$ which constitute a set of orthogonal functions. The coefficients of these polynomials are tabulated in math tables. See, for example, the *Handbook of Mathematical Functions*, Dover Publications. These polynomials are derived from the equations

$$C_k(x) = cos(kcos^{-1}x) \quad (|x| \le 1) \tag{11.39}$$

and

$$C_k(x) = cosh(kcosh^{-1}x) \quad |x| > 1 \tag{11.40}$$

From (11.39) with $k = 0$, we get

$$C_0(x) = cos(0cos^{-1}x) = 1 \tag{11.41}$$

With $k = 1$,

$$C_1(x) = \cos(1\cos^{-1}x) = x^{*} \tag{11.42}$$

With $k = 2$

$$C_2(x) = \cos(2\cos^{-1}x) = 2x^2 - 1 \tag{11.43}$$

and this is shown by letting $\cos^{-1}x = \alpha$. Then,

$$C_2(x) = \cos(2\alpha) = 2\cos^2\alpha - 1 = 2\cos^2(\cos^{-1}x) - 1$$

$$= 2\left[\underbrace{\cos(\cos^{-1}x)}_{x}\underbrace{\cos(\cos^{-1}x)}_{x} - 1\right] = 2x^2 - 1$$

We can also use MATLAB to convert these trigonometric functions to algebraic polynomials. For example,

```
syms x; expand(cos(2*acos(x)))
```

```
ans =
2*x^2-1
```

Using this iterated procedure we can show that with $k = 3, 4,$ and 5, we get

$$\begin{aligned} C_3(x) &= 4x^3 - 3x \\ C_4(x) &= 8x^4 - 8x^2 + 1 \\ C_5(x) &= 16x^5 - 20x^3 + 5x \end{aligned} \tag{11.44}$$

and so on.

We observe that for $k = even$, $C_k(x) = even$, and for $k = odd$, $C_k(x) = odd$.

The curves representing these polynomials are shown in Figure 11.17.

The Type I Chebyshev low-pass filter amplitude-square function is defined as

$$A^2(\omega) = \frac{\alpha}{1 + \varepsilon^2 C_k^2(\omega/\omega_C)} \tag{11.45}$$

The quantity ε^2 is a parameter chosen to provide the desired passband ripple and α is a constant chosen to determine the desired DC gain.

---

* We recall that if $x = \cos y$, then $y = \cos^{-1}x$, and $\cos y = x$.

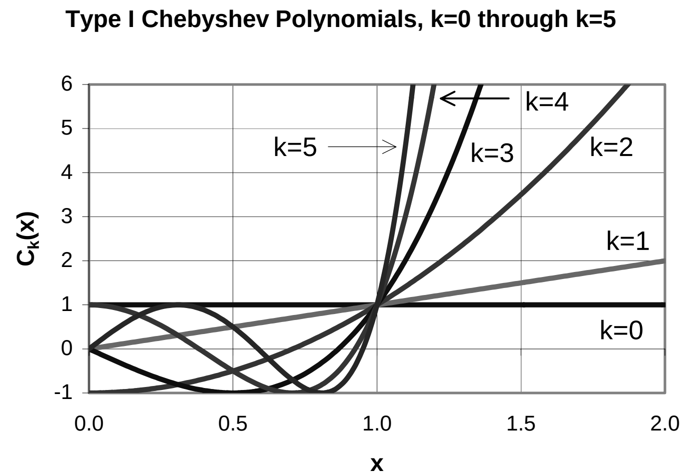

Figure 11.17. Type I Chebyshev polynomials

The parameter α in (11.45) is a constant representing the DC gain, ε is a parameter used in determining the ripple in the pass-band, the subscript k denotes both the degree of the Type I Chebyshev polynomial and the order of the transfer function, and ω_C is the cutoff frequency. This filter produces a sharp cutoff rate in the transition band.

Figure 11.18 shows Type I Chebyshev amplitude frequency responses for $k = 3$ and $k = 4$.

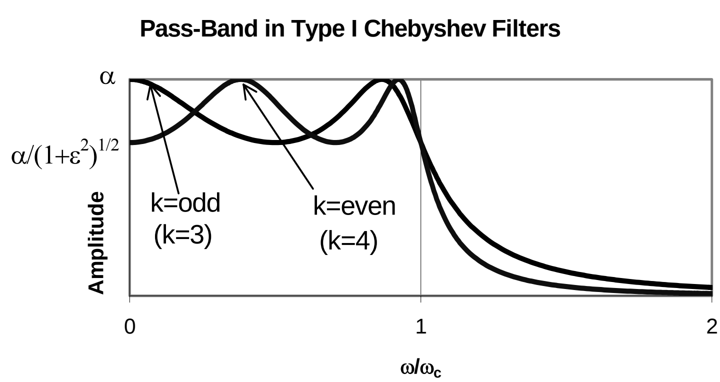

Figure 11.18. Chebyshev Type I Chebyshev low-pass filter for even and odd values of k.

The magnitude at $\omega = 0$ is α when $k = odd$ and $\alpha/\sqrt{1+\varepsilon^2}$ when $k = even$. This is shown in Figure 11.18. The cutoff frequency is the largest value of ω_C for which

$$A(\omega_C) = \frac{1}{\sqrt{1+\varepsilon^2}} \qquad (11.46)$$

Stated in other words, the pass-band is the range over which the ripple oscillates with constant bounds; this is the range from DC to ω_C. From (11.46), we observe that only when $\varepsilon = 1$ the magnitude at the cutoff frequency is 0.707 i.e., the same as in other types of filters. But when $0 < \varepsilon < 1$, the cutoff frequency is greater than the conventional $3\ dB$ cutoff frequency ω_C.

Table 11.4 gives the ratio of the conventional cutoff frequency $f_{3\ dB}$ to the ripple width frequency f_C of a Type I Chebyshev low-pass filter.

TABLE 11.4 Ratio of conventional cutoff frequency to ripple width frequency

Ratio of Conventional $f_{3\ dB}$ Cutoff Frequency to Ripple Width for Low-Pass Chebyshev Filters		
Ripple Width	$f_{3\ dB}/f_c$	
dB	k=2	k=4
0.1	1.943	1.213
0.5	1.390	1.093
1.0	1.218	1.053

The pass-band ripple r in dB, is defined as

$$r_{dB} = 10log_{10}\frac{A_{max}^2}{A_{min}^2} = 20log_{10}\frac{A_{max}}{A_{min}} \qquad (11.47)$$

where A_{max} and A_{min} are the maximum and minimum values respectively of the amplitude A in the pass-band interval. From (11.45),

$$A^2(\omega) = \frac{\alpha}{1+\varepsilon^2 C_k^2(\omega/\omega_C)} \qquad (11.48)$$

and A_{max}^2 occurs when $\varepsilon^2 C_k^2(\omega/\omega_C) = 0$. Then,

$$A_{max}^2 = \alpha \qquad (11.49)$$

To find A_{min}^2, we must first confirm that

$$C_k^2(\omega/\omega_C) \le 1$$

This can be shown to be true by (11.39), that is,

$$C_k(x) = cos(kcos^{-1}x) \quad |x| \le 1$$

or

$$|C_k(x)| \le 1 \quad for \quad -1 \le x \le 1$$

Therefore,

$$C_k^2(\omega / \omega_C)_{max} = 1$$

and

$$A_{min}^2 = \frac{\alpha}{1+\varepsilon^2} \tag{11.50}$$

Substitution of (11.49) and (11.50) into (11.47) yields

$$r_{dB} = 10log_{10}\frac{A_{max}^2}{A_{min}^2} = 10log_{10}\left[\frac{\alpha}{\alpha/(1+\varepsilon^2)}\right] = 10log_{10}(1+\varepsilon^2) \tag{11.51}$$

or

$$log_{10}(1+\varepsilon^2) = \frac{r_{dB}}{10}$$

or

$$1+\varepsilon^2 = 10^{r_{dB}/10}$$

or

$$\boxed{\varepsilon^2 = 10^{r_{dB}/10} - 1} \tag{11.52}$$

We have seen that when $k = odd$, there is a maximum at $\omega = 0$. At this frequency, (11.45) reduces to

$$A^2(0) = \alpha \tag{11.53}$$

and for a unity gain, $\alpha = 1$ when $k = odd$.

However, for unity gain when $k = even$, we must have $\alpha = 1+\varepsilon^2$. This is because at $\omega = 0$, we must have $C_k(0) = 1$ in accordance with(11.41). Then, the relation

$$A^2(\omega) = \frac{\alpha}{1+\varepsilon^2 C_k^2(\omega/\omega_C)}$$

reduces to

$$A^2(0) = \frac{\alpha}{1+\varepsilon^2 C_k^2(0)} = \frac{\alpha}{1+\varepsilon^2} = 1$$

or

$$\alpha = 1 + \varepsilon^2$$

For this choice of α, the amplitude response at maxima, corresponds to

$$A^2(\omega_{max}) = \frac{1+\varepsilon^2}{1+\varepsilon^2 C_k^2(\omega_{max}/\omega_C)}$$

and this will be maximum when

$$C_k^2(\omega_{max}/\omega_C) = 0$$

resulting in

$$A^2(\omega_{max}) = \frac{1+\varepsilon^2}{1} = 1+\varepsilon^2$$

or

$$A(\omega_{max}) = \sqrt{1+\varepsilon^2}$$

Example 11.10

Derive the transfer function $G(s)$ for the $k = 2$, Type I Chebyshev function that has pass-band ripple $r_{dB} = 1\ dB$, unity DC gain, and normalized cutoff frequency at $\omega_C = 1\ rad/s$.

Solution:

From (11.45),

$$A^2(\omega) = \frac{\alpha}{1+\varepsilon^2 C_k^2(\omega/\omega_C)} \tag{11.54}$$

and since $k = even$, for unity DC gain, we must have $\alpha = 1+\varepsilon^2$. Then, (11.54) becomes

$$A^2(\omega) = \frac{1+\varepsilon^2}{1+\varepsilon^2 C_k^2(\omega/\omega_C)}$$

For $k = 2$

$$C_2(x) = 2x^2 - 1$$

and

$$C_k^2(\omega/\omega_C) = C_k^2(\omega) = (2\omega^2 - 1)^2 = 4\omega^4 - 4\omega + 1$$

Also, from (11.52),

$$\varepsilon^2 = 10^{r_{dB}/10} - 1 = 10^{1/10} - 1 = 1.259 - 1 = 0.259$$

Then,

$$A^2(\omega) = \frac{1 + 0.259}{1 + 0.259(4\omega^4 - 4\omega + 1)} = \frac{1.259}{1.036\omega^4 - 1.036\omega^2 + 1.259}$$

and with $\omega^2 = -s^2$,

$$G(s)G(-s) = \frac{1.259}{1.036s^4 + 1.036s^2 + 1.259}$$

We find the poles from the roots of the denominator using MATLAB.

```
d=[1.036  0  1.036  0  1.259]; p=roots(d); fprintf(' \n'); disp('p1 = '); disp(p(1));...
disp('p2 = '); disp(p(2)); disp('p3 = '); disp(p(3)); disp('p4 = '); disp(p(4))
```

```
p1 =

  -0.5488 + 0.8951i

p2 =

  -0.5488 - 0.8951i

p3 =

   0.5488 + 0.8951i

p4 =

   0.5488 - 0.8951i
```

We now form the transfer function from the left half-plane poles $p_1 = -0.5488 + j0.8951$ and $p_2 = -0.5488 - j0.8951$. Then,

$$G(s) = \frac{K}{(s - p_1)(s - p_2)} = \frac{K}{(s + 0.5488 - j0.8951)(s + 0.5488 + j0.8951)}$$

We will use MATLAB to multiply the factors of the denominator.

```
syms s; den=(s+0.5488-0.8951*j)*(s+0.5488+0.8951*j); simple(expand(den))
```

```
ans =
s^2+686/625*s+22047709/20000000
```

```
686/625
```

```
ans =
    1.0976
```

```
22047709/20000000
```

```
ans =
    1.1024
```

Thus,

$$G(s) = \frac{K}{s^2 + 1.0976s + 1.1024}$$

and at $s = 0$,

$$G(0) = \frac{K}{1.1024}$$

Also, $A^2(0) = 1$, $A(0) = 1$

Then,

$$G(0) = A(0) = \frac{K}{1.1024} = 1$$

or

$$K = 1.1024$$

Therefore, the transfer function for Example 11.10 is

$$G(s) = \frac{1.1024}{s^2 + 1.0976s + 1.1024}$$

We can plot the attenuation band for Type I Chebyshev filters, as we did with the Butterworth filters in Figure 11.12, but we need to construct one for each value of dB in the ripple region. Instead, we will develop the following procedure.

We begin with the Chebyshev approximation

$$A^2(\omega) = \frac{\alpha}{1 + \varepsilon^2 C_k^2(\omega/\omega_C)} \tag{11.55}$$

and, for convenience, we let $\alpha = 1$. If we want the magnitude of this to be less than some value β for $\omega \ge \omega_C$, we should choose the value of k in $C_k^2(\omega/\omega_C)$ so that

$$\frac{1}{1 + \varepsilon^2 C_k^2(\omega/\omega_C)} \le \beta^2 \tag{11.56}$$

that is, we need to find a suitable value of the integer k so that (11.56) will be satisfied. As we have already seen from (11.52), the value of ε can be determined from

$$\varepsilon^2 = 10^{r_{dB}/10} - 1$$

once the band-pass ripple has been specified.

Next, we need to find $G(s)$ from

$$A^2(\omega) = G(s)G(-s)\big|_{s = j\omega}$$

and if we replace ω by s/j in (11.55) where $\alpha = 1$, we get

$$|G(s)|^2 = \frac{1}{1 + \varepsilon^2 C_k^2(s/j\omega_C)} \quad (11.57)$$

It can be shown that the poles of the left half of the s-plane are given by

$$s_i = \omega_C\left[-b\sin\frac{(2i+1)\pi}{2k} + jc\cos\frac{(2i+1)\pi}{2k}\right] \quad (11.58)$$

for $i = 0, 1, 2, \ldots, 2k-1$

The constants b and c in (11.58) can be evaluated from

$$b = \frac{m - m^{-1}}{2} \quad (11.59)$$

and

$$c = \frac{m + m^{-1}}{2} \quad (11.60)$$

where

$$m = (\sqrt{1 + \varepsilon^{-2}} + \varepsilon^{-1})^{1/k} \quad (11.61)$$

The transfer function is then computed from

$$G(s) = \frac{(-1)^k}{\prod_{i=0}^{k-1}\left(\frac{s}{s_i} - 1\right)} \quad (11.62)$$

Example 11.11

Design a Type I Chebyshev analog low-pass filter with $3\ dB$ band-pass ripple and $\omega_C = 5\ rad/s$. The attenuation for $\omega \geq 15\ \ rad/s$ must be at least $30\ dB/decade$.

Solution:

From (11.52),

$$\varepsilon^2 = 10^{r_{dB}/10} - 1 = 10^{3/10} - 1 = 1.9953 - 1 \approx 1$$

and the integer k must be chosen such that

$$10\log_{10}\frac{1}{1 + C_k^2(15/5)} \leq -30$$

or

$$-10\log_{10}(1 + C_k^2(3)) \le -30$$

$$-\log_{10}(1 + C_k^2(3)) \le -3$$

$$1 + C_k^2(3) \ge 10^3$$

To find the minimum value of k which satisfies this inequality, we compute the Chebyshev polynomials for $k = 0, 1, 2, 3, \ldots$. From (11.41) through (11.44), we get

$$C_0^2(3) = 1$$

$$C_1^2(3) = 3^2 = 9$$

$$C_2^2(3) = (2 \cdot 3^2 - 1)^2 = 17^2 = 289$$

$$C_3^2(3) = (4 \cdot 3^3 - 3 \cdot 3)^2 = 99^2 = 9801$$

and since $C_k^2(3)$ must be such that $1 + C_k^2(3) \ge 10^3$, we choose $k = 3$. Next, to find the poles of left half of the s-plane we first need to compute m, b, and c. From (11.61),

$$m = (\sqrt{1 + \varepsilon^{-2}} + \varepsilon^{-1})^{1/k} = \left(\sqrt{1 + \frac{1}{\varepsilon^2}} + \frac{1}{\sqrt{\varepsilon^2}}\right)^{1/3} = (\sqrt{2} + 1)^{1/3}$$

or

$$m = 1.3415$$

and

$$m^{-1} = 0.7454$$

Then, from (11.59) and (11.60),

$$b = \frac{1.3415 - 0.7454}{2} = 0.298$$

$$c = \frac{1.3415 + 0.7454}{2} = 1.043$$

and the poles for $i = 0, 1,$ and 2 are found from (11.58), that is

$$s_i = \omega_C\left[-b\sin\frac{(2i+1)\pi}{2k} + jc\cos\frac{(2i+1)\pi}{2k}\right]$$

Thus, the poles for this example are

$$s_0 = 5\left(-0.298\sin\frac{\pi}{6} + j1.043\cos\frac{\pi}{6}\right) = -0.745 + j4.516$$

$$s_1 = 5\left(-0.298\sin\frac{\pi}{2} + j1.043\cos\frac{\pi}{2}\right) = -1.49$$

$$s_2 = 5\left(-0.298\sin\frac{5\pi}{6} + j1.043\cos\frac{5\pi}{6}\right) = -0.745 - j4.516$$

Therefore, by substitution into (11.62) we get

$$G(s) = \frac{(-1)^3}{(s/s_0 - 1)(s/s_1 - 1)(s/s_2 - 1)} = \frac{-(-1.49)(-0.745 + j4.516)(-0.745 - j4.516)}{(s + 1.49)(s + 0.745 - j4.516)(s + 0.745 + j4.516)}$$

We will use MATLAB to do these computations.

```
-(-1.49)*(-0.745+j*4.516)*(-0.745-j*4.516)
```

```
ans =
   31.2144
```

```
syms s; den=(s+1.49)*(s+0.745-j*4.516)*(s+0.745+j*4.516); simple(expand(den))
```

```
ans =
s^3+149/50*s^2+23169381/1000000*s+3121442869/100000000
```

Then,

$$G(s) = \frac{31.214}{s^3 + 2.980s^2 + 23.169s + 31.214} \tag{11.63}$$

To verify that the derived transfer function $G(s)$ of (11.63) satisfies the filter specifications, we use the MATLAB code below to plot $|G(j\omega)|$.

```
w=0:0.01:100; s=j*w; Gs=31.214./(s.^3+2.98.*s.^2+23.169.*s+31.214);...
magGs=abs(Gs); dB=20.*log10(magGs); semilogx(w,dB); grid; hold on;
plot(w,magGs); grid; hold on; xlabel('Radian Frequency w rad/s');
ylabel('|G(w)| in dB'); title('Magnitude of G(w) versus Radian Frequency')
```

The plot is shown in Figure 11.19.

We can use the MATLAB **cheb1ap** function to design a Type I Chebyshev analog low-pass filter. Thus, the **[z,p,k] = cheb1ap(N,Rp)** statement where **N** denotes the order of the filter, returns the zeros, poles, and gain of an $N-th$ order normalized prototype Type I Chebyshev analog lowpass filter with ripple **Rp** decibels in the pass band.

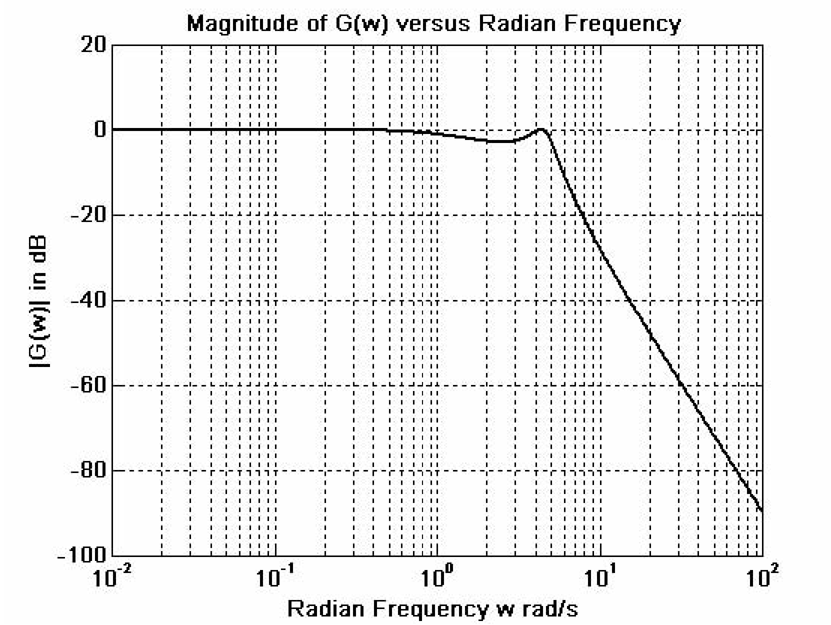

Figure 11.19. Plot for Example 11.11

Example 11.12

Use the MATLAB **cheb1ap** function to design a second order Type I Chebyshev low-pass filter with 3 dB ripple in the pass band.

Solution:

We use the code

```
w=0:0.05:400;                          % Define range to plot
[z,p,k]=cheb1ap(2,3);
[b,a]=zp2tf(z,p,k);                    % Convert zeros and poles of G(s) to polynomial form
[mag,phase]=bode(b,a,w); hold on
b                                      % Display the b coefficients
```

```
b =
          0          0     0.5012
```

```
a                                      % Display the a coefficients
```

```
a =
     1.0000     0.6449     0.7079
```

Now, with the known values of a and b we use the **bode** function to produce the Bode plots as follows.

```
bode(b,a), title('Bode Plot for Type 1 Chebyshev Low-Pass Filter')
```

This is shown in Figure 11.20.

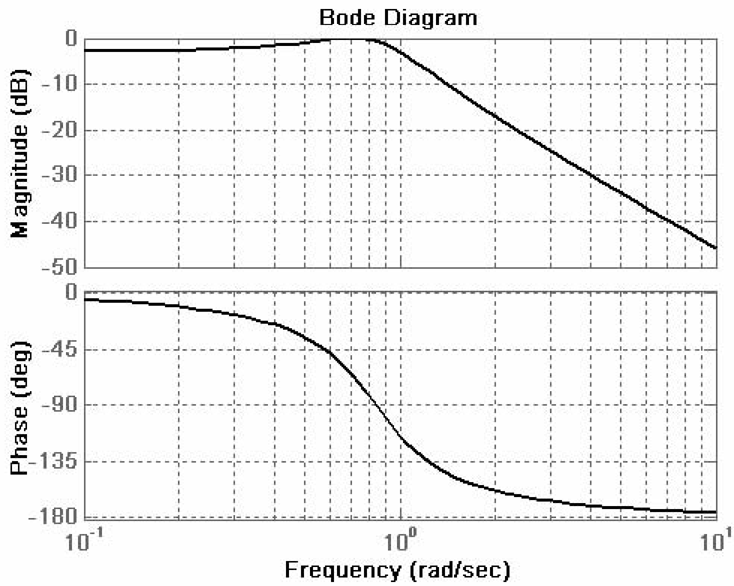

Figure 11.20. Bode plots for the filter of Example 11.12

On the Bode plots shown in Figure 11.20, the ripple is not so obvious. The reason is that this is a Bode plot with straight line approximations. To see the ripple, we use the following code:

```
w=0:0.01:10; [z,p,k]=cheb1ap(2,3); [b,a]=zp2tf(z,p,k); Gs=freqs(b,a,w);...
xlabel('Frequency in rad/s'), ylabel('Magnitude of G(s)'),...
semilogx(w,abs(Gs)); title('Type 1 Chebyshev Low-Pass Filter'), grid
```

The generated plot is shown in Figure 11.21.

11.6 Other Low-Pass Filter Approximations

We will briefly discuss two other filter types, the *Inverted Chebyshev*, and the *Cauer* or *Elliptic.*

The Inverted Chebyshev, also known as *Type II Chebyshev*, is characterized by the following amplitude-square approximation.

$$A^2(\omega) = \frac{\varepsilon^2 C_k^2(\omega_C/\omega)}{1+\varepsilon^2 C_k^2(\omega_C/\omega)} \tag{11.64}$$

and has the ripple in the stop-band as opposed to Type I which has the ripple in the pass-band. In (11.64), the frequency ω_C defines the beginning of the stop band.

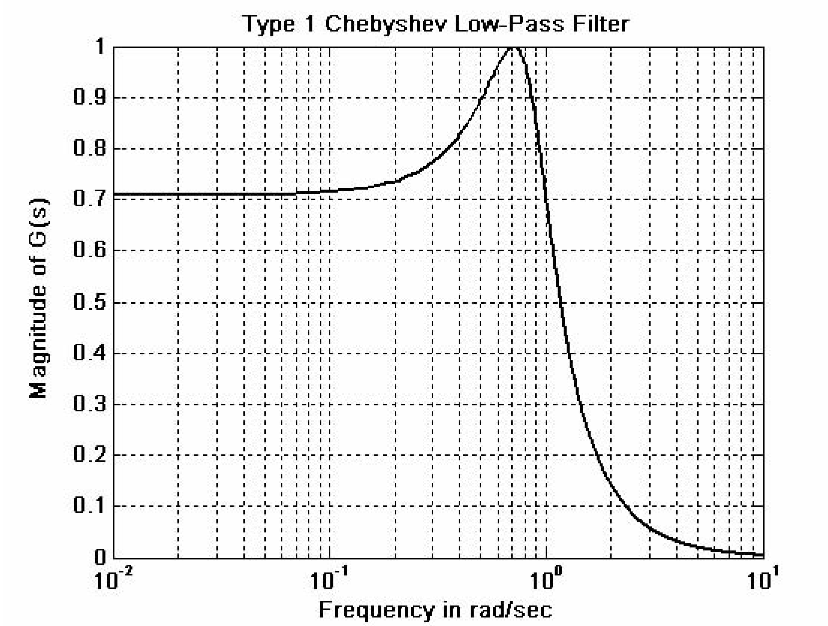

Figure 11.21. Amplitude characteristics for the filter of Example 11.12

The characteristics of a typical Type II Chebyshev low-pass filter are shown in Figure 11.22.

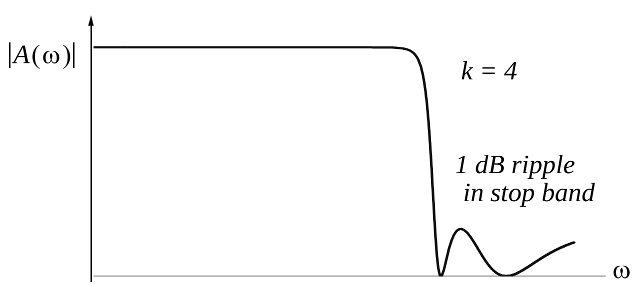

Figure 11.22. Type II Chebyshev low-pass filter

We can design Type II Chebyshev low-pass filters with the MATLAB **cheb2ap** function. Thus, the statement **[z,p,k] = cheb2ap(N,Rs)** where **N** denotes the order of the filter, returns the zeros, poles, and gain of an $N-th$ order normalized prototype Type II Chebyshev analog lowpass filter with ripple **Rs** decibels in the stop band.

Example 11.13

Using the MATLAB **cheb2ap** function, design a third order Type II Chebyshev analog filter with $3\ dB$ ripple in the stop band.

Solution:

We use the code

```
w=0:0.01:1000; [z,p,k]=cheb2ap(3,3); [b,a]=zp2tf(z,p,k); Gs=freqs(b,a,w);...
semilogx(w,abs(Gs)); xlabel('Frequency in rad/sec'); ylabel('Magnitude of G(s)');
title('Type 2 Chebyshev Low-Pass Filter, k=3, 3 dB ripple in stop band'); grid
```

The plot for this filter is shown in Figure 11.23.

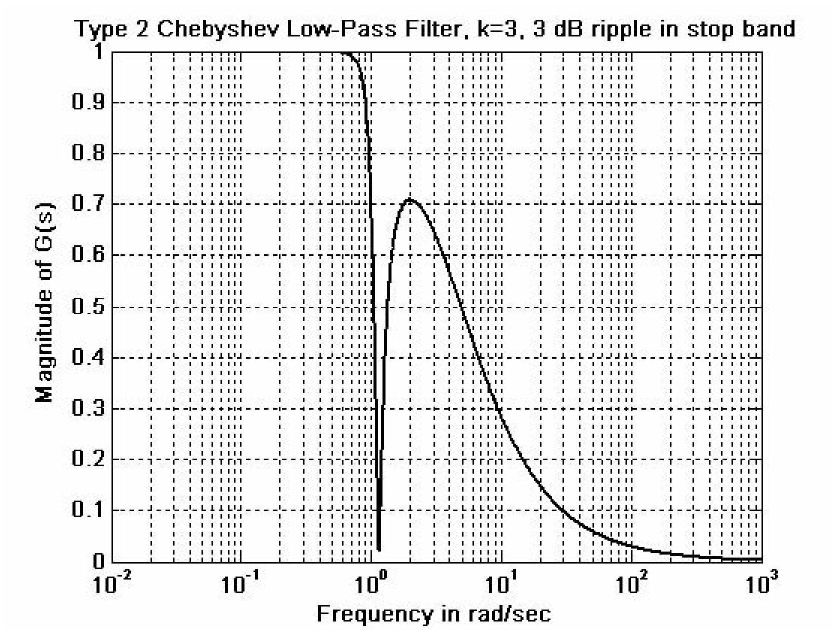

Figure 11.23. Plot for the filter of Example 11.13

The elliptic (Cauer) filters are characterized by the low-pass amplitude-squared function

$$A^2(\omega) = \frac{1}{1 + R_k^2(\omega/\omega_C)} \tag{11.65}$$

where $R_k(x)$ represents a rational elliptic function used with elliptic integrals. Elliptic filters have ripple in both the pass-band and the stop-band as shown in Figure 11.24.

We can design elliptic low-pass filters with the MATLAB **ellip** function. The statement **[b,a] = ellip(N,Rp,Rs,Wn,'s')** where **N** is the order of the filter, designs an $N-th$ order low-pass filter with ripple **Rp** decibels in the pass band, ripple **Rs** decibels in the stop band, **Wn** is the cutoff frequency, and **'s'** is used to specify analog elliptic filters. If **'s'** is not included in the above statement, MATLAB designs a digital filter. The plot of Figure 11.24 was obtained with the following MATLAB code:

```
w=0: 0.05: 500; [z,p,k]=ellip(5, 0.6, 20, 200, 's'); [b,a]=zp2tf(z,p,k);...
Gs=freqs(b,a,w); plot(w,abs(Gs)), title('5-pole Elliptic Low Pass Filter'); grid
```

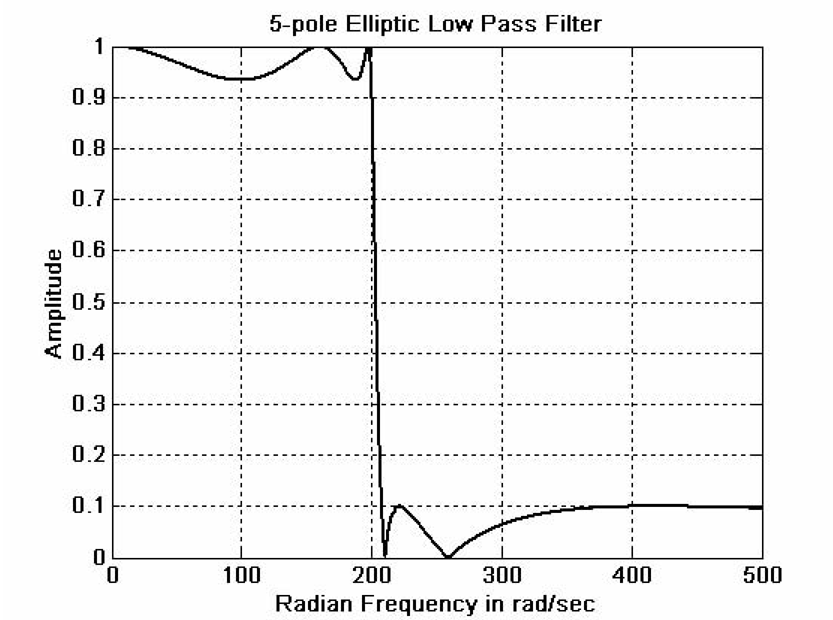

Figure 11.24. Characteristics of an elliptic low-pass filter

Example 11.14

Use MATLAB to design a four-pole elliptic analog low-pass filter with $0.5\ dB$ maximum ripple in the pass-band and $20\ dB$ minimum attenuation in the stop-band with cutoff frequency at $200\ rad/s$.

Solution:

The solution is obtained with the following MATLAB code.

```
w=0: 0.05: 500; [z,p,k]=ellip(4, 0.5, 20, 200, 's'); [b,a]=zp2tf(z,p,k);...
Gs=freqs(b,a,w); plot(w,abs(Gs)), title('4-pole Elliptic Low Pass Filter'); grid
```

The plot for this example is shown in Figure 11.25.

To form the transfer function $G(s)$, we need to know the coefficients a_i and b_i of the denominator and numerator respectively, of $G(s)$ in descending order. Because these are large numbers, we use the **format long** MATLAB command, and we get

```
format long
a
```

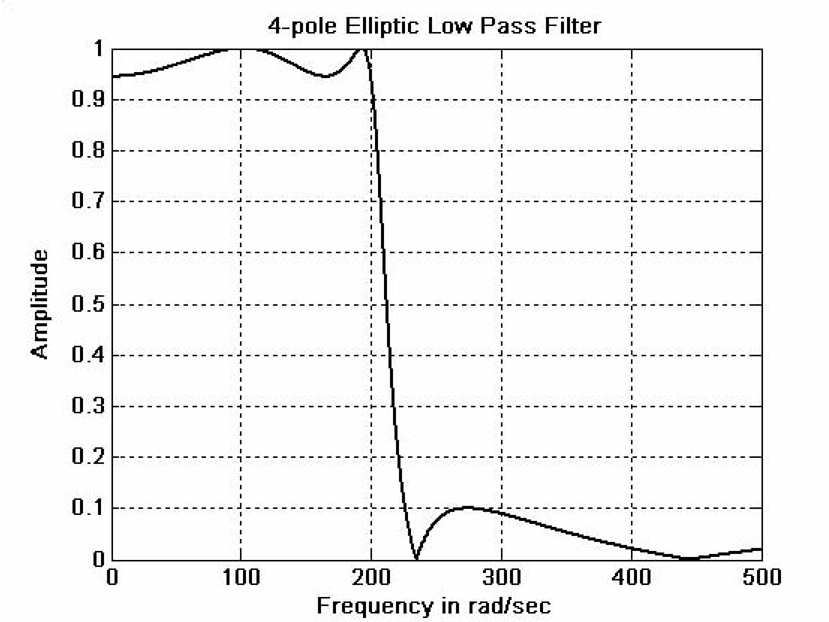

Figure 11.25. Plot for filter of Example 11.14

```
a =
  1.0e+009 *
  Columns 1 through 4
   0.00000000100000   0.00000033979343   0.00010586590805
      0.01618902078998
  Column 5
   2.07245309396647
```

b

```
b =
  1.0e+009 *
  Columns 1 through 4
   0.00000000010003     0      0.00003630258930     0
  Column 5
   2.04872991957189
```

Thus, the transfer function for this filter is

$$G(s) = \frac{2.0487 \times 10^9}{s^4 + 339.793s^3 + 105866s^2 + 16.189 \times 10^6 + 2.072 \times 10^9} \tag{11.66}$$

11.7 High-Pass, Band-Pass, and Band-Elimination Filters

Transformation methods have been developed where a low-pass filter can be converted to another type of filter simply by transforming the complex variable s. These transformations are listed in Table 11.5 where ω_C is the cutoff frequency of a low-pass filter. The procedure is best illustrated with the following examples.

TABLE 11.5 Filter transformations

Analog Filter Frequency Transformations	
Filter Type, Frequency	Replace s in $G(s)$ with
Low-Pass Filter, 3 dB pass-band, Normalized Frequency ω_C	No Change
Low-Pass Filter, 3 dB pass-band, Non-Normalized Frequency ω_{LP}	$\frac{s\omega_C}{\omega_{LP}}$
High-Pass Filter, 3 dB pass-band from $\omega = \omega_2$ to $\omega = \infty$	$\frac{\omega_{LP} \cdot \omega_2}{s}$
Band-Pass Filter, 3 dB pass-band from $\omega = \omega_{LP}$ to $\omega = \omega_2$	$\omega_C \cdot \frac{s^2 + \omega_{LP} \cdot \omega_2}{s(\omega_2 - \omega_{LP})}$
Band-Elimination Filter, 3 dB pass-band from $\omega = 0$ to $\omega = \omega_{LP}$, and from $\omega = \omega_2$ to $\omega = \infty$	$\omega_C \cdot \frac{s(\omega_2 - \omega_{LP})}{s^2 + \omega_{LP} \cdot \omega_2}$

Example 11.15

Compute the transfer function for a third-order band-pass Butterworth filter with $3\ dB$ pass-band from $3\ KHz$ to $5\ KHz$, from a third-order low-pass Butterworth filter with cutoff frequency $f_C = 1\ KHz$.

Solution:

We first find the transfer function for a third-order Butterworth low-pass filter with normalized frequency $\omega_C = 1\ rad/s$. Using the MATLAB function **buttap** we write and execute the following code:

```
[z, p, k]=buttap(3); [b,a]=zp2tf(z,p,k)
b =
     0      0      0      1
a =
    1.0000    2.0000    2.0000    1.0000
```

Thus, the transfer function for the third-order Butterworth low-pass filter with normalized cutoff frequency $\omega_c = 1\ rad/s$ is

$$G(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} \tag{11.67}$$

Next, the actual cutoff frequency is given as $f_C = 1\ KHz$ or $\omega_C = 2\pi \times 10^3\ rad/s$. Accordingly, in accordance with Table 11.5, we replace s with

$$\frac{s\omega_C}{\omega_{LP}} = \frac{s}{2\pi \times 10^3}$$

and we get

$$\begin{aligned} G\left(\frac{s}{2\pi \times 10^3}\right) = G'(s) &= \frac{1}{(s/(2\pi \times 10^3))^3 + 2(s/(2\pi \times 10^3))^2 + 2(s/(2\pi \times 10^3)) + 1} \\ &= \frac{2.48 \times 10^{11}}{s^3 + 1.26 \times 10^4 s^2 + 7.89 \times 10^7 s + 2.48 \times 10^{11}} \end{aligned} \tag{11.68}$$

Now, we replace s in the last expression of (11.68) with

$$\omega_C \cdot \frac{s^2 + \omega_{LP} \cdot \omega_2}{s(\omega_2 - \omega_{LP})} \tag{11.69}$$

or

$$1 \cdot \frac{s^2 + 2\pi \times 10^3 \times 3 \times 2\pi \times 10^3}{s(3 \times 2\pi \times 10^3 - 2\pi \times 10^3)} = \frac{s^2 + 12 \times \pi^2 \times 10^6}{s(4\pi \times 10^3)} = \frac{s^2 + 1.844 \times 10^8}{1.257 \times 10^4 s}$$

Then,

$$G''(s) = \frac{2.48 \times 10^{11}}{\left(\frac{s^2 + 1.844 \times 10^8}{1.257 \times 10^4 s}\right)^3 + \left(\frac{s^2 + 1.844 \times 10^8}{1.257 \times 10^4 s}\right)^2 + \frac{s^2 + 1.844 \times 10^8}{1.257 \times 10^4 s} + 2.48 \times 10^{11}}$$

We see that the computations, using the transformations of Table 11.5 become quite tedious. Fortunately, we can use the MATLAB **lp2lp**, **lp2hp**, **lp2bp**, and **lp2bs** functions to transform a low pass filter with normalized cutoff frequency, to another low-pass filter with any other specified frequency, or to a high-pass filter, or to a band-pass filter, or to a band elimination filter respectively.

Example 11.16

Use the MATLAB **buttap** and **lp2lp** functions to find the transfer function of a third-order Butterworth low-pass filter with cutoff frequency $f_C = 2\ KHz$.

Solution:

We will use the **buttap** command to find the transfer function $G(s)$ of the filter with normalized cutoff frequency at $\omega_C = 1\ rad/s$. Then, we will use the command **lp2lp** to transform $G(s)$ to $G'(s)$ with cutoff frequency at $f_C = 2\ KHz$, or $\omega_C = 2\pi \times 2 \times 10^3\ rad/s$.

```
format short e
% Design 3 pole Butterworth low-pass filter (wcn=1 rad/s)
[z,p,k]=buttap(3);
[b,a]=zp2tf(z,p,k);          % Compute num, den coefficients of this filter (wcn=1rad/s)
f=1000:1500/50:10000;        % Define frequency range to plot
w=2*pi*f;                    % Convert to rads/sec
fc=2000;                     % Define actual cutoff frequency at 2 KHz
wc=2*pi*fc;                  % Convert desired cutoff frequency to rads/sec
[bn,an]=lp2lp(b,a,wc);       % Compute num, den of filter with fc = 2 kHz
Gsn=freqs(bn,an,w);          % Compute transfer function of filter with fc = 2 kHz
semilogx(w,abs(Gsn)); grid; hold on; xlabel('Radian Frequency w (rad/sec)'),...
ylabel('Magnitude of Transfer Function'),...
title('3-pole Butterworth low-pass filter with fc=2 kHz or wc = 12.57 kr/s')
```

The plot for the magnitude of this transfer function is shown in Figure 11.26.

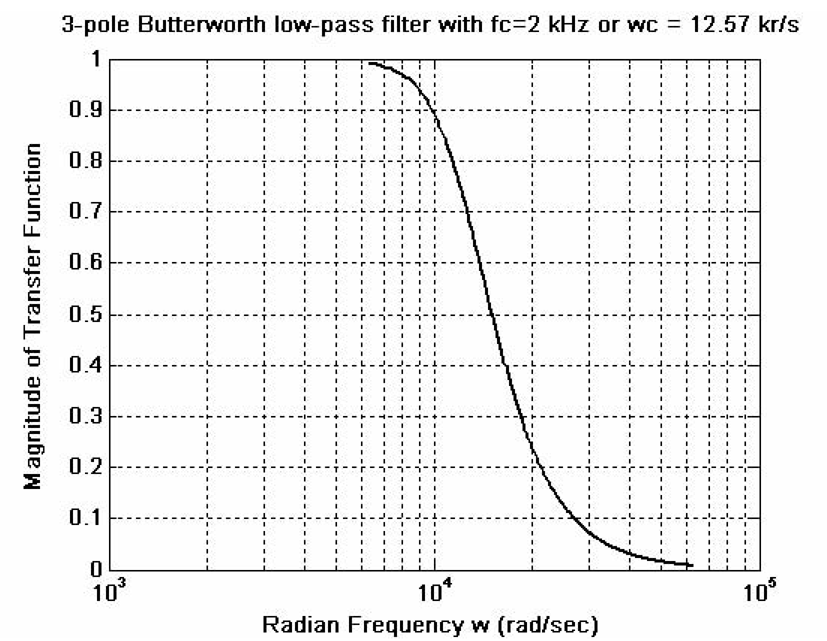

Figure 11.26. Magnitude for the transfer function of (11.71)

The coefficients of the numerator and denominator of the transfer function are as follows:

b

```
b =
     0     0     0     1
```

```
a
```

```
a =
  1.0000e+000  2.0000e+000  2.0000e+000  1.0000e+000
```

```
bn
```

```
bn =
       1.9844e+012
an
```

```
an =
  1.0000e+000  2.5133e+004  3.1583e+008  1.9844e+012
```

Thus, the transfer function with normalized cutoff frequency $\omega_{Cn} = 1\ rad/s$ is

$$G(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} \tag{11.70}$$

and with actual cutoff frequency $\omega_{Cn} = 2\pi \times 2000\ rad/s = 1.2566 \times 10^4$ is

$$G'(s) = \frac{1.9844 \times 10^{12}}{s^3 + 2.5133 \times 10^4 s^2 + 3.1583 \times 10^8 s + 1.9844 \times 10^{12}} \tag{11.71}$$

Example 11.17

Use the MATLAB commands **cheb1ap** and **lp2hp** to find the transfer function of a 3-pole Type I Chebyshev high-pass analog filter with cutoff frequency $f_C = 5\ KHz$.

Solution:

We will use the **cheb1ap** command to find the transfer function $G(s)$ of the low-pass filter with normalized cutoff frequency at $\omega_C = 1\ rad/s$. Then, we will use the command **lp2hp** to transform $G(s)$ to another $G'(s)$ with cutoff frequency at $f_C = 5\ KHz$ or $\omega_C = 2\pi \times 5 \times 10^3\ rad/s$

```
% Design 3 pole Type 1 Chebyshev low-pass filter, wcn=1 rad/s
[z,p,k]=cheb1ap(3,3);
[b,a]=zp2tf(z,p,k);            % Compute num, den coef. with wcn=1 rad/s
f=1000:100:100000;             % Define frequency range to plot
fc=5000;                       % Define actual cutoff frequency at 5 KHz
wc=2*pi*fc;                    % Convert desired cutoff frequency to rads/sec
[bn,an]=lp2hp(b,a,wc);         % Compute num, den of high-pass filter with fc = 5 KHz
Gsn=freqs(bn,an,2*pi*f);       % Compute and plot transfer function of filter with fc = 5 KHz
semilogx(f,abs(Gsn)); grid; hold on
```

```
xlabel('Frequency (Hz)'); ylabel('Magnitude of Transfer Function')
title('3-pole Type 1 Chebyshev high-pass filter with fc=5 KHz ')
```

The magnitude of this transfer function is plotted as shown in Figure 11.27.

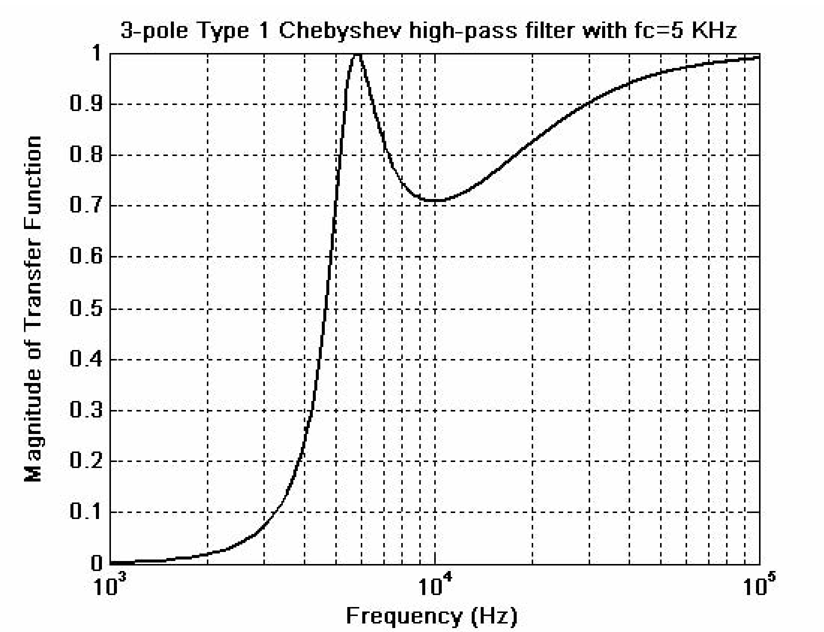

Figure 11.27. Magnitude of the transfer function of (11.73) for Example 11.17

The coefficients of the numerator and denominator of the transfer function are as follows:

```
b

b =
     0            0            0   2.5059e-001

a

a =
   1.0000e+000  5.9724e-001  9.2835e-001  2.5059e-001

bn

bn =
   1.0000e+000  2.2496e-011 -1.4346e-002 -6.8973e-003

an

an =
   1.0000e+000  1.1638e+005  2.3522e+009  1.2373e+014
```

Therefore, the transfer function with normalized cutoff frequency $\omega_{Cn} = 1\ rad/s$ is

$$G(s) = \frac{0.2506}{s^3 + 0.5972s^2 + 0.9284s + 0.2506} \qquad (11.72)$$

and with actual cutoff frequency $\omega_{Cn} = 2\pi \times 5000\ rad/s = 3.1416 \times 10^4$, is

$$G'(s) = \frac{s^3}{s^3 + 1.1638 \times 10^5 s^2 + 2.3522 \times 10^9 s + 1.2373 \times 10^{14}} \qquad (11.73)$$

Example 11.18

Use the MATLAB functions **buttap** and **lp2bp** to find the transfer function of a 3-pole Butterworth analog band-pass filter with the pass band frequency centered at $f_0 = 4\ KHz$, and bandwidth $BW = 2\ KHz$.

Solution:

We will use the **buttap** function to find the transfer function $G(s)$ of the low-pass filter with normalized cutoff frequency at $\omega_C = 1\ rad/s$. We found this transfer function in Example 11.15 as given by (11.67). However, to maintain a similar MATLAB code as in the previous examples, we will include it in the code that follows. Then, we will use the command **lp2bp** to transform $G(s)$ to another $G'(s)$ with centered frequency at $f_0 = 4\ KHz$ or $\omega_0 = 2\pi \times 4 \times 10^3\ rad/s$, and bandwidth $BW = 2\ KHz$ or $BW = 2\pi \times 2 \times 10^3\ rad/s$

```
format short e
[z,p,k]=buttap(3);          % Design 3 pole Butterworth low-pass filter with wcn=1 rad/s
[b,a]=zp2tf(z,p,k);         % Compute numerator and denominator coefficients for wcn=1
rad/s
f=100:100:100000;           % Define frequency range to plot
f0=4000;                    % Define centered frequency at 4 KHz
W0=2*pi*f0;                 % Convert desired centered frequency to rads/sec
fbw=2000;                   % Define bandwidth
Bw=2*pi*fbw;                % Convert desired bandwidth to rads/sec
[bn,an]=lp2bp(b,a,W0,Bw);% Compute num, den of band-pass filter
% Compute and plot the magnitude of the transfer function of the band-pass filter
Gsn=freqs(bn,an,2*pi*f); semilogx(f,abs(Gsn)); grid; hold on
xlabel('Frequency f (Hz)'); ylabel('Magnitude of Transfer Function');
title('3-pole Butterworth band-pass filter with f0 = 4 KHz, BW = 2KHz')
```

The plot for this band-pass filter is shown in Figure 11.28.

The coefficients b_n and a_n are as follows:

```
bn

bn =
```

```
1.9844e+012  -4.6156e+001  -1.6501e+005  -2.5456e+009
```

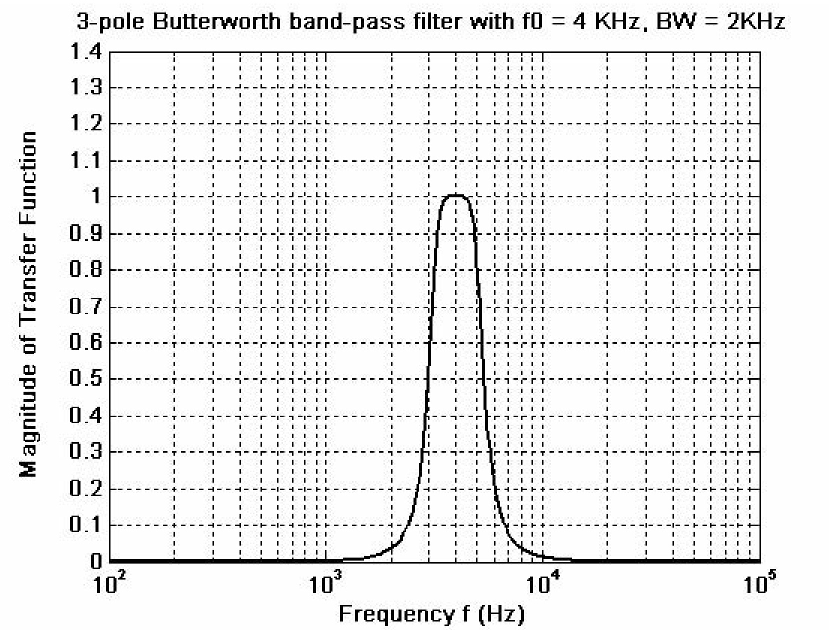

Figure 11.28. Plot for the band-pass filter of Example 11.18

```
an

an =
  1.0000e+000  2.5133e+004  2.2108e+009  3.3735e+013  1.3965e+018
  1.0028e+022  2.5202e+026
```

Since the numerator b_n and denominator a_n coefficients are too large to be written in a one line equation, we have listed them in tabular form as shown below.

Power of s	Numerator b_n	Denominator a_n
s^6	0	1
s^5	0	2.5133×10^4
s^4	0	2.2108×10^9
s^3	1.9844×10^{12}	3.3735×10^{13}
s^2	-4.6156×10^1	1.3965×10^{18}
s	-1.6501×10^5	1.0028×10^{22}
Constant	-2.5456×10^9	2.5202×10^{26}

Example 11.19

Use the MATLAB functions **buttap** and **lp2bs** to find the transfer function of a 3-pole Butterworth band-elimination (band-stop) filter with the stop band frequency centered at $f_0 = 5\ KHz$, and bandwidth $BW = 2\ KHz$.

Solution:

We will use the **buttap** function to find the transfer function $G(s)$ of the low-pass filter with normalized cutoff frequency at $\omega_C = 1\ rad/s$. We found this transfer function as (11.67) in Example 11.15. However, to maintain a similar MATLAB code as in the previous examples, we will include it in the code which follows. Accordingly, we will use the **lp2bs** function to transform $G(s)$ to another transfer function $G'(s)$ with centered frequency at $f_0 = 5\ KHz$, or $\omega_0 = 2\pi \times 5 \times 10^3\ rad/s$, and bandwidth $BW = 2\ KHz$ or $BW = 2\pi \times 2 \times 10^3\ rad/s$.

```
[z,p,k]=buttap(3);          % Design 3-pole Butterworth low-pass filter, wcn = 1 r/s
[b,a]=zp2tf(z,p,k);         % Compute num, den coefficients of this filter, wcn=1 r/s
f=100:100:100000;           % Define frequency range to plot
f0=5000;                    % Define centered frequency at 5 kHz
W0=2*pi*f0;                 % Convert centered frequency to r/s
fbw=2000;                   % Define bandwidth
Bw=2*pi*fbw;                % Convert bandwidth to r/s
% Compute numerator and denominator coefficients of desired band stop filter
[bn,an]=lp2bs(b,a,W0,Bw);
% Compute and plot magnitude of the transfer function of the band stop filter
Gsn=freqs(bn,an,2*pi*f); semilogx(f,abs(Gsn)); grid; hold on
xlabel('Frequency in Hz'); ylabel('Magnitude of Transfer Function');
title('3-pole Butterworth band-elimination filter with f0=5 KHz, BW = 2 KHz')
```

The amplitude response of this band elimination filter is shown in Figure 11.29.

The coefficients b_n and a_n are as follows:

```
bn

bn =
  1.0000e+000   -7.6352e-012   2.9609e+009   -1.5071e-002
  2.9223e+018   -7.4374e+006   9.6139e+026

an

an =
  1.0000e+000    2.5133e+004   3.2767e+009    5.1594e+013
  3.2340e+018    2.4482e+022   9.6139e+026
```

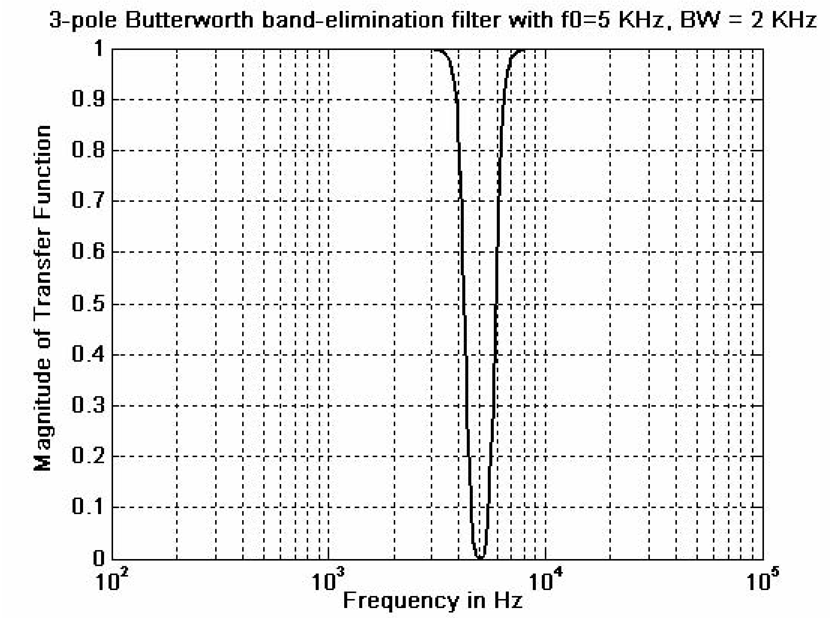

Figure 11.29. Amplitude response for the band-elimination filter of Example 11.19

As in the previous example, we list the numerator b_n and denominator a_n coefficients in tabular form as shown below.

Power of s	Numerator b_n	Denominator a_n
s^6	1	1
s^5	-7.6352×10^{-12}	2.5133×10^{4}
s^4	2.9609×10^{-6}	3.2767×10^{9}
s^3	-1.5071×10^{-2}	5.1594×10^{13}
s^2	2.9223×10^{18}	3.2340×10^{18}
s	-7.4374×10^{6}	2.4482×10^{22}
Constant	9.6139×10^{26}	9.6139×10^{26}

In all of the above examples, we have shown the magnitude, but not the phase response of each filter type. However, we can use the MATLAB function **bode(num,den)** to generate both the magnitude and phase responses of any transfer function describing the filter type, as shown by the following example.

Example 11.20

Use the MATLAB **bode** function to plot the magnitude and phase characteristics of the 3-pole Butterworth low-pass filter with unity gain and normalized frequency at $\omega_C = 1\ rad/s$.

Solution:

We know, from Example 11.15, that the transfer function for this type of filter is

$$G(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

We can obtain the magnitude and phase characteristics with the following MATLAB code:

```
num=[0 0 0 1]; den=[1 2 2 1]; bode(num,den),...
title('Bode Plot for 3-pole Butterworth Low-Pass Filter'); grid
```

The magnitude and phase characteristics are shown in Figure 11.30.

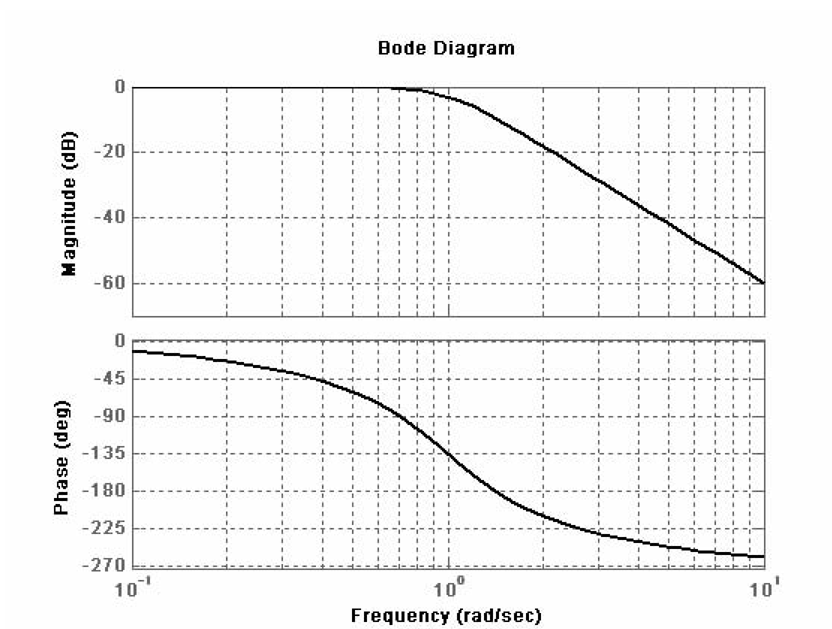

Figure 11.30. Bode plots for 3-pole Butterworth low-pass filter

We conclude the discussion on analog filters with Table 11.6 listing the advantages and disadvantages of each type.

TABLE 11.6 Advantages / Disadvantages of different types of filters

Filter Type	Advantages	Disadvantages
Butterworth	• Simplest design • Flat pass band	• Slow rate of attenuation for order 4 or less
Chebyshev Type 1	• Sharp cutoff rate in transition (pass to stop) band	• Ripple in pass band • Bad (non-linear) phase response
Chebyshev Type II	• Sharp cutoff rate in transition (pass to stop) band	• Ripple in stop band • Bad (non-linear) phase response
Elliptic (Cauer)	• Sharpest cutoff rate among all other types of filters	• Ripple in both pass band and stop band • Worst (most non-linear) phase response among the other types of filters.