Series 2 solution of mathematical logic tutorial

Exercise 1 Solution :

1- The formal system that produces theorems **kst, kstst, ksststst,..........** from the axiom **k** is :

- The set of alphabet $\Sigma = \{k, s, t\}$.
- The set **W** which represents all the words generated by this formal system as well as the axioms used. Such us $\mathbf{W} = \{k, kst, kstst, ksststst, kstststst, ...\}$.
- The set of axioms $\mathbf{A} = \{k\}$.
- The set of deduction rules **R** which contains a single rule r_1 such as $\mathbf{R} = \{r_1 : x \longrightarrow xst | x \in \mathbf{W}\}$.

2-The formal system that produces theorems **ca, caba, cababa, cabababa, ...** from the axiom **c** is :

- The set of alphabet $\Sigma = \{a, b, c\}$.
- The set **W** $\{c, ca, caba, cababa, cabababa, ...\}$.
- The set of axioms $\mathbf{A} = \{c\}$.
- The set of deduction rules **R** that contains two rules r_1 and r_1 such as $\mathbf{R} = \{r_1 : c \longrightarrow ca, r_2 : cx \longrightarrow cxba\}$ such as x is any sequence $\in \{a, b, c\}$ and x $\neq \epsilon$.

3- The formal system that produces theorems **b, ba, baa, baaa, baaaa,.....** from the axiom **b** is :

- The set of alphabet $\Sigma = \{a, b\}$.
- The set **W** $\{b, ba, baa, baaa, baaaa, ...\}$.
- The set of axioms $\mathbf{A} = \{b\}$.
- The set of deduction rules $\mathbf{R} = \{r_1 : bx \longrightarrow bxa\}$

Exercise 2 Solution :

1- To prove that **MUIUI** is a theorem, we are going to use the derivation tree so to arrive at **MUIUI** which is the leaf of the tree we are looking for, we just have to do (starting with the axiom) :
$MI \xrightarrow{R2} MII \xrightarrow{R2} MIIII \xrightarrow{R3} MUI \xrightarrow{R2} MUIUI$ **True**.

2- **UM** is not a theorem because from the axiom **MI**we cannot reach **UM** that is to say : The axiom **MI** contains **M** at the beginning and all the rules allow you to generate **M** at the beginning according to the axiom, i.e. no rule allows you to rewrite the **M** (example : which has the form $M \longrightarrow ...$).

3- **MU** is not a theorem and we can prove this using another theorem-producing system well known as arithmetic.
The sequence **MU** contains zero (**I**) which is a multiple of 3 and the axiom at the beginning **MI** contains a single (**I**) therefore a number of (**I**) not a multiple of 3. So we have to look for a way to eliminate the **I** based on the inference rules.

We notice that the rules **R1** and **R4** do not change the number of **I**. The rule **R3** decreases the number of **I** by 3, so it doesn't change it unless it is divisible by 3. The rule **R2** doubles the number of **I**. Since 2n can only be divided by 3 if n is divisible by 3 and the rule **R2** does not produce a multiple of 3, therefore no rule produces a multiple of 3.
Conclusion : **MU** is not a theorem
Note : With the derivation tree, the system **MIU** is not decidable for **MU**.

Exercise 3 Solution :
1- For the sequence //p/q/// yes we can.
Proof :
using the derivation tree, we can have : pq $\xrightarrow{a}$ /pq/ $\xrightarrow{a}$ //pq// $\xrightarrow{b}$ //p/q///.

2- We cannot find /p//q/, because according to the rules of the p-q system, we always have theorems which have the following form : r(///**...**//)**p** s(//**...**/)**q** r+s(///**...**///).

For the sequence //////p///q///////// yes we can.
Proof :
pq $\xrightarrow{a}$ /pq/ $\xrightarrow{a}$ //pq// $\xrightarrow{a}$ ///pq/// $\xrightarrow{a}$ ////pq//// $\xrightarrow{a}$ /////pq///// $\xrightarrow{a}$ //////pq////// $\xrightarrow{b}$ //////p/q/////// $\xrightarrow{b}$ //////p//q//////// $\xrightarrow{b}$ //////p///q/////////.

Exercise 4 Solution :
1- D $\xrightarrow{a}$ DC Yes.

2- D $\xrightarrow{a}$ DC $\xrightarrow{a}$ DCC $\xrightarrow{a}$ DCCC Yes.

3- D $\xrightarrow{b}$ ADA $\xrightarrow{b}$ AADAA $\xrightarrow{b}$ AAADAAA Yes.

4- D $\xrightarrow{a}$ DC $\xrightarrow{a}$ DCC $\xrightarrow{a}$ DCCC $\xrightarrow{b}$ ADCCCA $\xrightarrow{a}$ ADCCCAC $\xrightarrow{a}$ ADCCCACC $\xrightarrow{b}$ AADCCCACCA $\xrightarrow{c}$ AADCCCABCA $\xrightarrow{c}$ AADCCCABBA Yes.

Exercise 5 Solution :
Q_1-
1- a^4bc^4 is a theorem.
Proof :
From the set A of axioms we can generate the axiom $A_1 = a^3bc$ by replacing the i in the set A by 1.
By applying the rule R_1 using the axiom A_1, i.e. we replace the first element of the rule by A_1 and the second ($R_{1)} = (A_1$,A_1)) we will have :
$(a^3bc,\, a^3bc) \xrightarrow{R_1} a^4bc^4$. o a^4bc^4 is a theorem.

2- a^6bc^6 is a theorem.

Proof :
- First, we create the first axiom $A_1 = a^5bc^3$ by replacing in the general form of the axioms the i by 2.
-We also create a second axiom $A_2 = a^3bc$ by replacing i in the general form of the axioms with 1.
By applying the rule $R_1 = (a^5bc^3,\, a^3bc) \xrightarrow{R_1} a^6bc^6$. So a^6bc^6 is a theorem.

3- a^5bc^5 is not a theorem.
Proof :
Because if we add two axioms according to the rule R_1, we always find an even number of i (number of a = the number of c) and if we add a theorem generated with an even number of i and an axiom , we will never find equality between a and c.
Q_2-
To find the possible forms of theorems, we must try to apply the inference rule by changing the elements of this rule (incoming data).
1- The first validated form is the form of axioms, i.e. $forme_1 = \{a^{2i+1}bc^{2i-1} | i \geq 1\}$.
From the axioms, we can apply rule R_1, that is :
$(a^{2i+1}bc^{2i-1} | i \geq 1, a^{2s+1}bc^{2s-1} | s \geq 1) \xrightarrow{R_1} (a^{2(i+s)}bc^{2(i+s)}) \longrightarrow \{a^{2k}bc^{2k} | k \geq 2\}$.
We can also have :
$(a^{2k}bc^{2k} | k \geq 2, a^{2i+1}bc^{2i-1} | i \geq 1) \xrightarrow{R_1} (a^{2(k+i)-1}bc^{2(k+i)+1}) \longrightarrow \{a^{2p-1}bc^{2p+1} | p \geq 3\}$.
Or :
$(a^{2i+1}bc^{2i-1} | i \geq 1, a^{2k}bc^{2k} | k \geq 2) \xrightarrow{R_1} (a^{2(i+k)+1}bc^{2(i+k)-1}) \longrightarrow \{a^{2p+1}bc^{2p-1} | p \geq 3\} \subset \{a^{2i+1}bc^{2i-1} | i \geq 1\}$.
Or :
$(a^{2k}bc^{2k} | k \geq 2, a^{2p-1}bc^{2p+1} | p \geq 3) \xrightarrow{R_1} (a^{2g+1}bc^{2g-1}) \subset \{a^{2i+1}bc^{2i-1}\}$.
And so on.
So after several tests, we conclude that the general form of theorems is as follows : $\{a^{2i+1}bc^{2i-1} | i \geq 1\} \cup \{a^{2k}bc^{2k} | k \geq 2\} \cup \{a^{2p-1}bc^{2p+1} | p \geq 3\} \cup \{a^{2r+2}bc^{2r-2} | p \geq 4\} \cup \{a^{2r-2}bc^{2r+2} | p \geq 4\}$,... etc.