Lecture 2 - Frequency Selective Filters

2.1 Applications

Low-pass : to extract short-term average or to eliminate high-frequency fluctuations (eg. noise filtering, demodulation, etc.)

High-pass : to follow small-amplitude high-frequency perturbations in presence of much larger slowly-varying component (e.g. recording the electrocardiogram in the presence of a strong breathing signal)

Band-pass : to select a required modulated carrier frequency out of many (e.g. radio)

Band-stop : to eliminate single-frequency (e.g. mains) interference (also known as notch filtering)

2.2 Design of Analogue Filters

We will start with an analysis of analogue low-pass filters, since a low-pass filter can be mathematically transformed into any other standard type.

Design of a filter may start from consideration of

- The desired frequency response.
- The desired phase response.

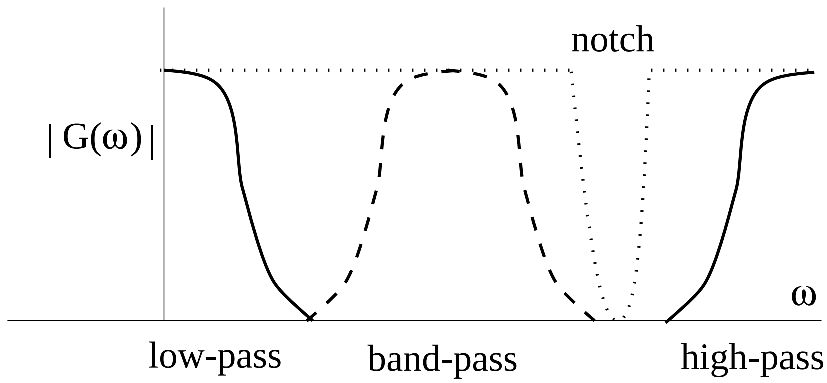

Figure 2.1: *Standard filters.*

The majority of the time we will consider the first case. Consider some desired response, in the general form of the (squared) magnitude of the transfer function, i.e. $|G(s)|^2$. This response is given as

$$|G(s)|^2 = G(s)G^*(s)$$

where $*$ denotes complex conjugation. If $G(s)$ represents a *stable* filter (its poles are on the LHS of the s-plane) then $G^*(s)$ is *unstable* (as its poles will be on the RHS).

The design procedure consists then of

- Considering some desired response $|G(s)|^2$ as a polynomial in *even* powers of s.
- Designing the filter with the stable part of $G(s), G^*(s)$.

This means that, for any given filter response in the positive frequency domain, a mirror image exists in the negative frequency domain.

2.2.1 Ideal low-pass filter

Any frequency-selective filter may be described either by its frequency response (more common) or by its impulse response. The narrower the band of frequencies transmitted by a filter, the more extended in time is its impulse response waveform. Indeed, if the support in the frequency domain is *decreased* by a factor of a (i.e. made narrower) then the required support in the time domain is *increased* by a factor of a (you should be able to prove this).

Consider an *ideal* low-pass filter with a "brick wall" amplitude cut-off and no phase shift, as shown in Fig. 2.2.

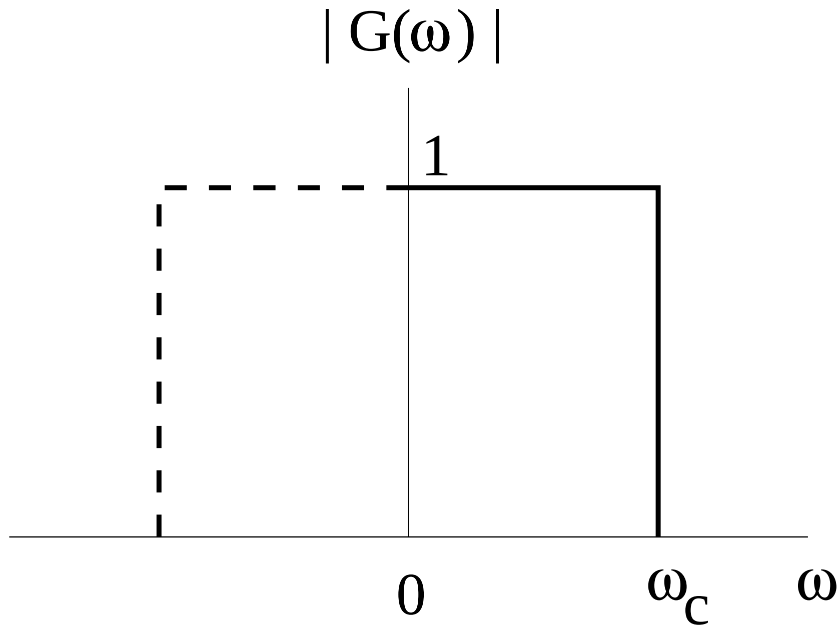

Figure 2.2: *The ideal low-pass filter. Note the requirement of response in the negative frequency domain.*

Calculate the impulse response as the inverse Fourier transform of the frequency response:

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1 \cdot e^{j\omega t}d\omega = \frac{1}{2\pi jt}(e^{j\omega_c t} - e^{-j\omega_c t})$$

hence,

$$g(t) = \frac{\omega_c}{\pi}(\frac{\sin \omega_c t}{\omega_c t})$$

Figure 2.3 shows the impulse response for the filter (this is also referred to as the *filter kernel*).

The output starts infinitely long before the impulse occurs – i.e. the filter is not realisable in *real time*.

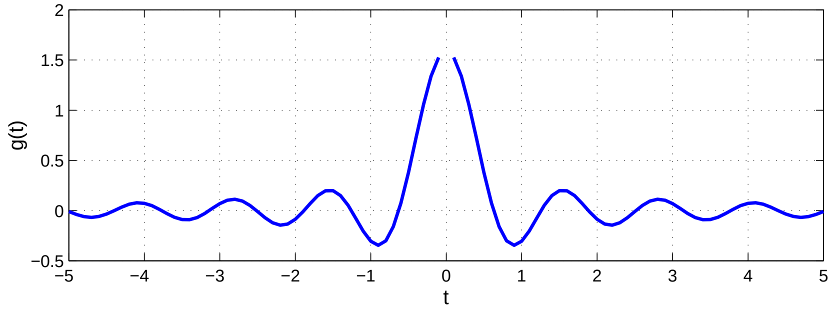

Figure 2.3: *Impulse response (filter kernel) for the ILPF. The zero crossings occur at integer multiples of* π/ω_c.

A delay of time T such that

$$g(t) = \frac{\omega_c}{\pi} \frac{\sin \omega_c (t - T)}{\omega_c (t - T)}$$

would ensure that *most* of the response occurred after the input (for large T). The use of such a delay, however, introduces a phase lag proportional to frequency, since $arg\{G(j\omega\} = \omega T$. Even then, the filter is still not exactly realisable; instead the design of analogue filters involves the choice of the most suitable *approximation* to the ideal frequency response.

2.3 Practical Low-Pass Filters

Assume that the low-pass filter transfer function $G(s)$ is a rational function in s. The types of filters to be considered in the next few pages are *all-pole designs*, which means that $G(s)$ will be of the form:

$$G(s) = \frac{1}{(a_n s^n + a_{n-1} s^{n-1} + \ldots a_1 s + a_0)}$$

$$\text{or} G(j\omega) = \frac{1}{(a_n (j\omega)^n + a_{n-1} (j\omega)^{n-1} + \ldots a_1 j\omega + a_0)}$$

The *magnitude-squared response* is $|G(j\omega)|^2 = G(j\omega) \cdot G(-j\omega)$. The denominator of $|G(j\omega)|^2$ is hence a polynomial in *even* powers of ω. Hence the task of approximating the ideal magnitude-squared characteristic is that of choosing a suitable denominator polynomial in ω^2, i.e. selecting the function H in the following expression:

$$|G(j\omega)|^2 = \frac{1}{1 + H\{(\frac{\omega}{\omega_c})^2\}}$$

where ω_c = nominal cut-off frequency and H = rational function of $(\frac{\omega}{\omega_c})^2$.

The choice of H is determined by functions such that $1 + H\{(\omega/\omega_c)^2\}$ is close to unity for $\omega < \omega_c$ and rises rapidly after that.

2.4 Butterworth Filters

$$H\{(\frac{\omega}{\omega_c})^2\} = \{(\frac{\omega}{\omega_c})^2\}^n = (\frac{\omega}{\omega_c})^{2n}$$

i.e.

$$|G(j\omega)|^2 = \frac{1}{1 + (\frac{\omega}{\omega_c})^{2n}}$$

where n is the order of the filter. Figure 2.4 shows the response on linear (a) and log (b) scales for various orders n.

2.4.1 Butterworth filter – notes

1. $|G| = \frac{1}{\sqrt{2}}$ for $\omega = \omega_c$ (i.e. magnitude response is 3dB down at cut-off frequency)

2. For large n:
   - in the region $\omega < \omega_c$, $|G(j\omega)| = 1$
   - in the region $\omega > \omega_c$, the steepness of $|G|$ is a direct function of n.

3. Response is known as *maximally flat*, because

$$\left.\frac{d^n G}{d\omega^n}\right|_{\omega=0} = 0 \quad \text{for} \quad n = 1, 2, \ldots 2N-1$$

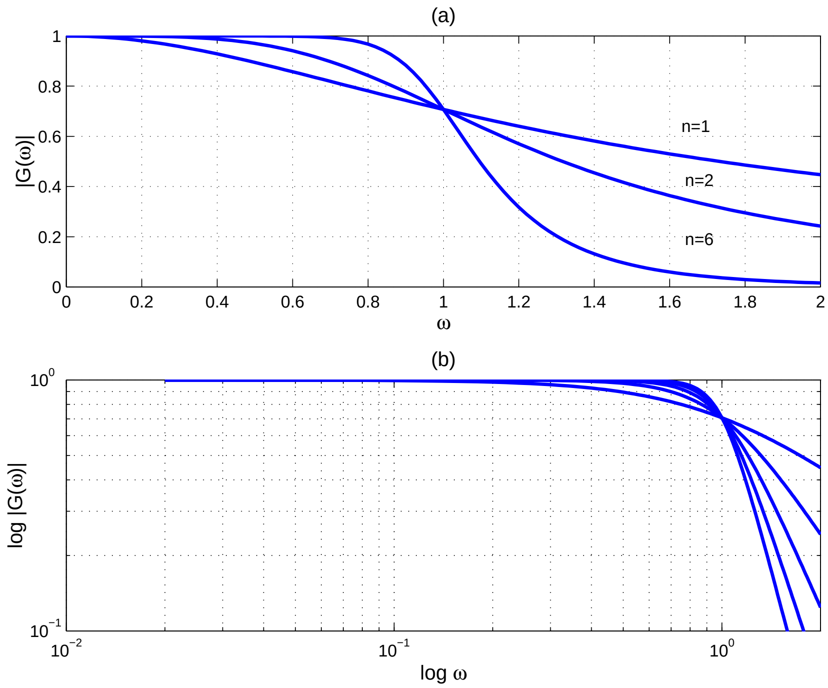

Figure 2.4: *Butterworth filter response on (a) linear and (b) log scales. On a log-log scale the response, for $\omega > \omega_c$ falls off at approx -20db/decade.*

Proof

Express $|G(j\omega)|$ in a binomial expansion:

$$|G(j\omega)| = \{1 + (\frac{\omega}{\omega_c})^{2n}\}^{-\frac{1}{2}} = 1 - \frac{1}{2}(\frac{\omega}{\omega_c})^{2n} + \frac{3}{8}(\frac{\omega}{\omega_c})^{4n} - \frac{5}{16}(\frac{\omega}{\omega_c})^{6n} + \ldots$$

It is then easy to show that the first $2n - 1$ derivatives are all zero at the origin.

2.4.2 Transfer function of Butterworth low-pass filter

$$|G(j\omega)| = \sqrt{G(j\omega)G(-j\omega)}$$

Since $G(j\omega)$ is derived from $G(s)$ using the substitution $s \to j\omega$, the reverse operation can also be done, i.e. $\omega \to -js$

$$\sqrt{G(s)G(-s)} = \frac{1}{\sqrt{1+(-j\frac{s}{\omega_c})^{2n}}}$$

$$\text{or} \qquad G(s)G(-s) = \frac{1}{1+(\frac{-js}{\omega_c})^{2n}}$$

Thus the *poles* of

$$\frac{1}{1+(\frac{-js}{\omega_c})^{2n}}$$

belong either to $G(s)$ or $G(-s)$. The poles are given by:

$$(\frac{-js}{\omega_c})^{2n} = -1 = e^{j(2k+1)\pi}, \quad k = 0, 1, 2, \ldots, 2n-1$$

Thus

$$\frac{-js}{\omega_c} = e^{j(2k+1)\frac{\pi}{2n}}$$

Since $j = e^{j\frac{\pi}{2}}$, then we have the final result:

$$s = \omega_c \exp j[\frac{\pi}{2} + (2k+1)\frac{\pi}{2n}]$$

i.e. the poles have the same modulus ω_c and equi-spaced arguments. For example, for a fifth-order Butterworth low-pass filter (LPF), $n = 5$:

$$\frac{\pi}{2n} = 18^o \rightarrow \frac{\pi}{2} + (2k+1)\frac{\pi}{2n} = 90^o + (18^o, 54^o, 90^o, 126^o, \text{etc}\ldots)$$

i.e. the poles are at:

$$\underbrace{108^o, 144^o, 180^o, 216^o, 252^o,}_{\text{in L.H. s-plane therefore stable}} \qquad \underbrace{288^o, 324^o, 360^o, 396^o, 432^o}_{\text{in R.H.S. s-plane therefore unstable}}$$

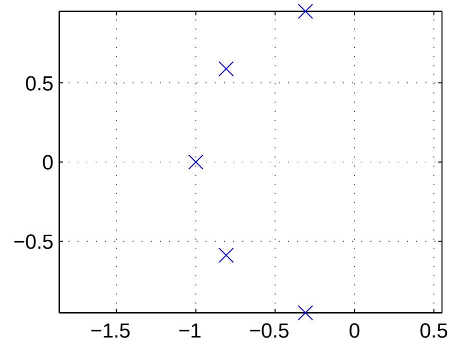

Figure 2.5: *Stable poles of 5–th order Butterworth filter.*

We want to design a *stable* filter. Since each unstable pole is $(-1)\times$ a stable pole, we can let the stable ones be in $G(s)$, and the unstable ones in $G(-s)$.

Therefore the poles of $G(s)$ are $\omega_c e^{j108^o}, \omega_c e^{j144^o}, \omega_c e^{j180^o}, \omega_c e^{j216^o}, \omega_c e^{j252^o}$ as shown in Figure 2.5. Hence,

$$G(s) = \frac{1}{1-\frac{s}{p_k}} = \frac{1}{(1+(\frac{s}{\omega_c}))\{1+2\cos 72^o\frac{s}{\omega_c}+(\frac{s}{\omega_c})^2\}\{1+2\cos 36^o\frac{s}{\omega_c}+(\frac{s}{\omega_c})^2\}}$$

or, multiplying out:

$$G(s) = \frac{1}{1+3.2361\frac{s}{\omega_c}+5.2361(\frac{s}{\omega_c})^2+5.2361(\frac{s}{\omega_c})^3+3.2361(\frac{s}{\omega_c})^4+(\frac{s}{\omega_c})^5}$$

Note that the coefficients are “palindromic” (read the same in reverse order) – this is true for all Butterworth filters. Poles are always on same radii, at $\frac{\pi}{n}$ angular spacing, with “half-angles” at each end. If n is odd, one pole is real.

n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1	1.0000							
2	1.4141	1.0000						
3	2.0000	2.0000	1.0000					
4	2.6131	3.4142	2.6131	1.0000				
5	3.2361	5.2361	5.2361	3.2361	1.0000			
6	3.8637	7.4641	9.1416	7.4641	3.8637	1.0000		
7	4.4940	10.0978	14.5918	14.5918	10.0978	4.4940	1.0000	
8	5.1258	13.1371	21.8462	25.6884	21.8462	13.1371	5.1258	1.0000

<u>Butterworth LPF coefficients for $a \leq 8$</u>

2.4.3 Design of Butterworth LPFs

The only design variable for Butterworth LPFs is the order of the filter n. If a filter is to have an attenuation A at a frequency ω_a:

$$|G|^2_{\omega_a} = \frac{1}{A^2} = \frac{1}{1 + (\frac{\omega_a}{\omega_c})^{2n}}$$

$$\text{i.e.} \quad n = \frac{\log(A^2 - 1)}{2 \log \frac{\omega_a}{\omega_c}}$$

$$\text{or since usually } A \gg 1, \qquad n \approx \frac{\log A}{\log \frac{\omega_a}{\omega_c}}$$

Butterworth design – example

Design a Butterworth LPF with at least 40 dB attenuation at a frequency of 1kHz and 3dB attenuation at f_c = 500Hz.

Answer

40dB $\rightarrow$ A = 100; $\quad \omega_a = 2000\pi$ and $\omega_c = 1000\pi$ rads/sec

$$\text{Therefore } n \approx \frac{\log_{10} 100}{\log_{10} 2} = \frac{2}{0.301} = 6.64$$

Hence n = 7 meets the specification

Check: Substitute $s = j2$ into the transfer function from the above table for $n = 7$

$$|G(j2)| = \frac{1}{|87.38 - 93.54j|}$$

which gives $A = 128$.

2.5 Equiripple Filters

You may have noticed that the Butterworth response is monotone i.e. it has no *ripple*. If a certain amount of ripple is allowed in the pass-band and/or the stop-band, filters which have a sharper cut-off than the Butterworth filter (for a given order n) can be designed. There are three types of *equi-ripple* filters:

- Chebyshev Type I (equi-ripple in the pass-band)
- Chevyshev Type II (also known as inverse Chebyshev – equi-ripple in the stop-band)
- Elliptic (equi-ripple on both pass-band and stop-band)

2.5.1 Chebyshev polynomials

The n-order Chebyshev polynomial $T_n(x)$ can be expressed as:

$$T_n(x) = \cos(n \cos^{-1} x)$$

for $0 \leq x \leq 1$; and

$$T_n(x) = \cosh(n \cosh^{-1} x)$$

for $x > 1$.

These two expressions are equivalent in that the latter can be derived from the former and so the result is a single Chebyshev polynomial which applies for $0 \leq x \leq \infty$.

Alternatively, $T_n(x)$ can be expressed as a polynomial in x, which can be evaluated for any x:

$$T_1(x) = \cos(\cos^{-1} x) = x$$

$$T_2(x) = \cos 2\theta \text{ where } \theta = \cos^{-1} x$$

$$\text{i.e. } T_2(x) = 2\cos^2\theta - 1 = 2x^2 - 1$$

Obviously, the same results are also valid with the $\cosh$ function.

Generally,

$$T_{n+1}(x) + T_{n-1}(x) = \cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta = 2xT_n(x)$$

Thus:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

is a recurrence relationship which allows the computation of $T_{n+1}(x)$ from the two previous polynomials.

Using the recurrence relationship,

$$T_3(x) = 2xT_2(x) - T_1(x) = 2x(2x^2-1) - x = 4x^3 - 3x$$

$$T_4(x) = 2xT_3(x) - T_2(x) = 2x(4x^3-3x) - 2x^2 - 1 = 8x^4 - 8x^2 + 1$$

The graphs of Fig. 2.6 below show that, for large n, $T_n(x)$ diverges very rapidly for $|x| > 1$; for example, $T_4(1.5) = 23.5$. This is just the kind of behaviour we need for a good filter function.

For a Chebyshev Type I filter (i.e. equi-ripple in the pass-band):

$$H\{(\frac{\omega}{\omega_c})^2\} = \varepsilon^2 T_n^2(\frac{\omega}{\omega_c})$$

$$\text{ie.}|G(j\omega)|^2 = \frac{1}{1+\varepsilon^2 Tn^2(\frac{\omega}{\omega_c})}$$

Chebyshev type I – notes

1. The parameter $\varepsilon(0 < \varepsilon < 1)$ sets the ripple amplitude in the *ripple pass-band* which is defined as $0 \leq \omega \leq \omega_c$. Since, for $\omega \leq \omega_c, 0 \leq T_n^2 \leq 1$, $|G(j\omega)|$ will fluctuate between 1 and $\frac{1}{\sqrt{1+\varepsilon^2}}$.

   e.g. with $\varepsilon = 0.5088$, the amplitude will vary between 1 and $\frac{1}{\sqrt{1.26}}$, ie. 0.89 or 1dB ripple since $20\log_{10} 0.89 = -1.0$.

2. At $\omega = 0$, $\quad T_n^2(x) = \{\cos(n\cos^{-1} 0)\}^2$

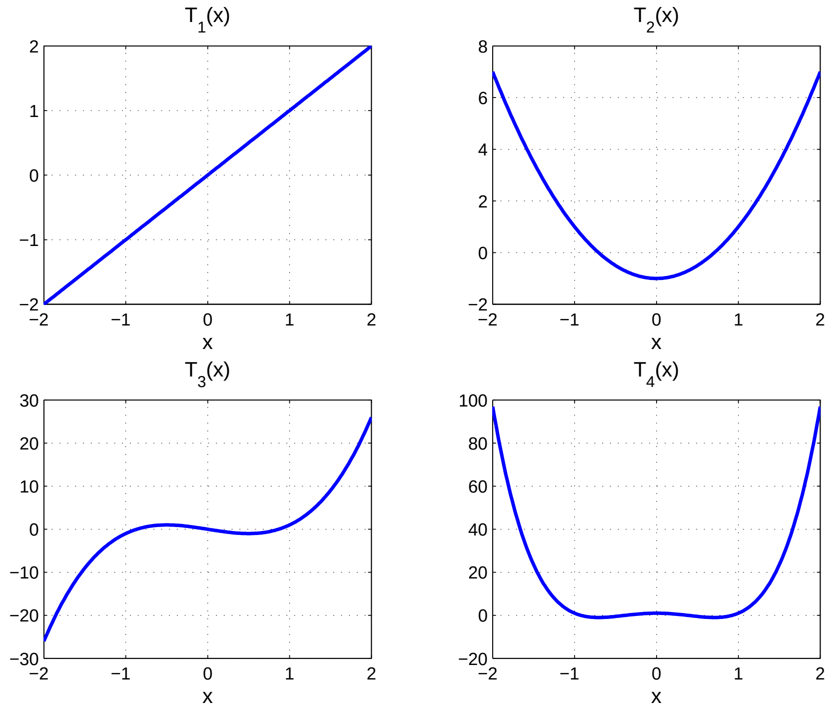

Figure 2.6: *Chebyshev functions.*

ie. $T_n^2(x) = 0$ if n is odd and 1 if n is even

3. ω_c, in this case, is *not* the $3dB$ cut-off frequency.

   By definition, ω_{3dB} is given by:

$$\frac{1}{1+\varepsilon^2 T_n^2(\frac{\omega_{3dB}}{\omega_c})} = \frac{1}{2}$$

   ie. $\varepsilon^2 T_n^2(\frac{\omega_{3dB}}{\omega_c}) = 1$ which means that $\omega_{3dB} > \omega_c$, since $\varepsilon < 1$

4. For $\omega > \omega_c, T_n^2$ gets large and $|G(j\omega)|$ decreases monotonically, as for the Butterworth LPF, but much more rapidly.

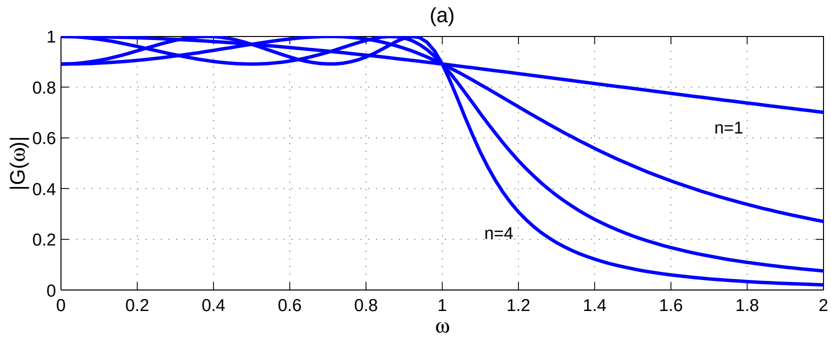

Figure 2.7: *Response of Chebyshev type I filter. The lower limit of ripple in the pass-band is* $1/\sqrt{1+\varepsilon^2}$. *Here* $\varepsilon = 0.5088$.

2.5.2 Transfer function of Chebyshev Type I LPF

As before, put $|G(j\omega)| = \sqrt{G(j\omega)G(-j\omega)}$, and let $\omega \to -js$

$$G(s)G(-s) = \frac{1}{1+\varepsilon^2 Tn^2(\frac{-js}{\omega_c})}$$

The poles are given by $T_n(\frac{-js}{\omega_c}) = \pm j\varepsilon^{-1}$. Of the $2n$ roots, assign the stable ones to $G(s)$.

Example

$n = 3; \varepsilon = \frac{1}{2}$

$$T_3(\frac{-js}{\omega_c}) = 4(\frac{-js}{\omega_c})^3 - 3(\frac{-js}{\omega_c}) = j4(\frac{s}{\omega_c})^3 + j \cdot 3\frac{s}{\omega_c} = \pm j\varepsilon^{-1} = \pm j2$$

i.e. 2 cubics

$$\underbrace{4(\frac{s}{\omega_c})^3 + 3(\frac{s}{\omega_c}) - 2 = 0}_{roots\,\frac{s}{\omega_c}=+\frac{1}{2},-\frac{1}{4}\pm j\sqrt{\frac{15}{16}}} \text{ and } \underbrace{4(\frac{s}{\omega_c})^3 + 3(\frac{s}{\omega_c}) + 2 = 0}_{roots\,\frac{s}{\omega_c}=-\frac{1}{2},+\frac{1}{4}\pm j\sqrt{\frac{15}{16}}}$$

Select as stable poles $s = -\frac{1}{2}\omega_c, s = (-\frac{1}{4} \pm j\sqrt{\frac{15}{16}})\omega_c$

Therefore $\quad G(s) = \prod_{k=1}^{3} \frac{1}{1-\frac{s}{p_k}} = \frac{1}{\{1+2\frac{s}{\omega_c}\}\{1+\frac{1}{2}\frac{s}{\omega_c} + (\frac{s}{\omega_c})^2\}}$ (factorised form)

(Note $\zeta = \frac{1}{4}$ for complex pair – it was $\frac{1}{2}$ for Butterworth filter)

$$\text{Hence} \quad G(s) = \frac{1}{1 + 2\frac{1}{2}(\frac{s}{\omega_c}) + 2(\frac{s}{\omega_c})^2 + 2(\frac{s}{\omega_c})^3} \quad \text{(polynomial form)}$$

In general, the poles lie on an *ellipse* (this is only meaningful for $n \geq 5$; if $n \leq 4$, an ellipse can be drawn through any *set* of poles).

A large ε means a larger ripple, narrower ellipse and less damped complex poles, ie. a more oscillatory impulse or step response *but* a steeper cut-off.

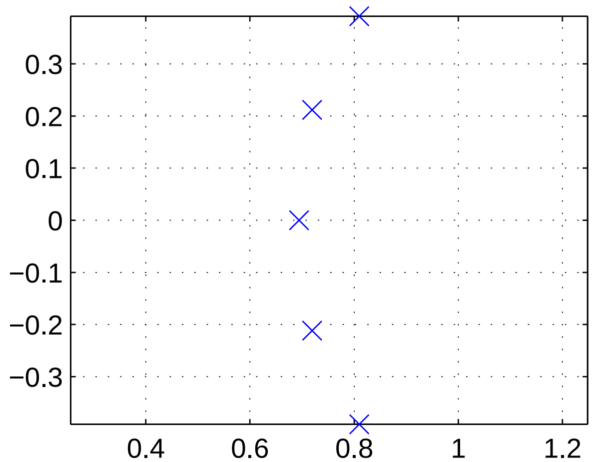

Figure 2.8: *Stable poles of 5–th order Chebyshev Type I filter.*

2.5.3 Design of Chebyshev Type I LPFs

If a filter is to have an attenuation A at a frequency ω_a:

$$|G|^2_{\omega_a} = \frac{1}{A^2} = \frac{1}{1 + \varepsilon^2 T_n^2(\frac{\omega_a}{\omega_c})}$$

$$\text{ie.} \quad A^2 - 1 = \varepsilon^2 T_n^2(\frac{\omega_a}{\omega_c}) = \varepsilon^2 \cosh^2(n \cosh^{-1} \frac{\omega_a}{\omega_c})$$

$$\text{i.e.} \quad \cosh(n \cosh^{-1} \frac{\omega_a}{\omega_c}) = \frac{\sqrt{A^2 - 1}}{\varepsilon}$$

$$\text{or} \quad n = \frac{\cosh^{-1} \frac{\sqrt{A^2-1}}{\varepsilon}}{\cosh^{-1} \frac{\omega_a}{\omega_c}}$$

Example

Design a Chebyshev Type I LPF to meet the following specifications:

- Maximum ripple of 1dB in pass-band from 0 to 50Hz
- 3dB cut-off frequency at f = 65 Hz
- Attenuation of at least 40 dB for $f \geq 250$ Hz

Answer

1. As shown on page 21, 1dB ripple corresponds to a value of 0.5088 for ε.
2. **3dB cut-off**: $\omega_{3dB} = 130\pi$ rads/sec. Minimum n given by:

$$n = \frac{\cosh^{-1} \frac{1}{0.5088}}{\cosh^{-1} 1.3} = 1.714$$

Therefore $n \geq 2$ meets the 3 dB cut-off requirement.

3. For 40dB attenuation, $A = 100$. Minimum n given by:

$$n = \frac{cosh^{-1}\sqrt{(\frac{10^4-1}{)}0.5088}}{\cosh^{-1} 5} = 2.606$$

ie. $n = 3$ meets the overall specification.

2.5.4 Chebyshev Type II low-pass filters

These exhibit equiripple behaviour in the *stop-band* but have the same behaviour as a Butterworth filter (ie. maximally flat around $\omega = 0$ in the pass-band. This behaviour cannot be achieved with an all-pole filter; Chebyshev Type II filters have a transfer function $G(s)$ which includes zeroes on the imaginary axis as well as poles in the left-half s-plane.

$$|G(j\omega)|^2 = \frac{1}{1 + \varepsilon^2 \frac{T_n^2(\frac{\omega_r}{\omega_c})}{T_n^2(\frac{\omega_r}{\omega})}}$$

where ω_r = lowest frequency at which stop-band loss attains a specified value.

The magnitude-squared response for Chebyshev Type I and II filters in Fig. 2.9. For both these filters, the pass-band edge is at $\omega = \omega_c$ where $|G|^2 = \frac{1}{1+\varepsilon^2}$ and the stop-band edge is at $\omega = \omega_r$ where $|G|^2 = \frac{1}{A^2}$.

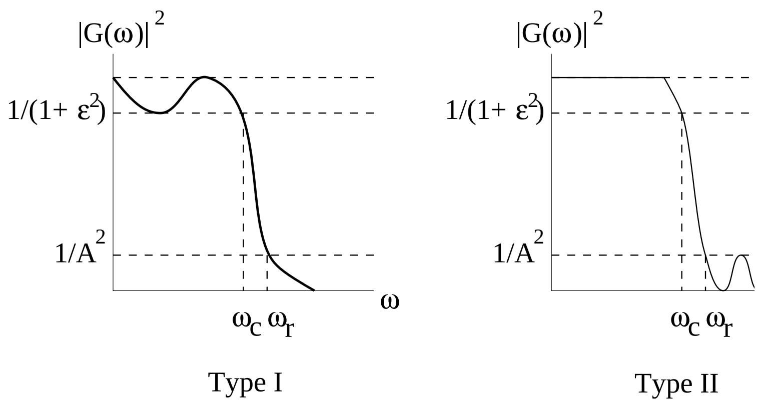

Figure 2.9: *Response of Chebyshev type I, II filters.*

Thus, for $n = 5$:

$$G(s) = \frac{\{1 + (\frac{s}{\omega_1})^2\}\{1 + (\frac{s}{\omega_2})^2\}}{1 + a_1(\frac{s}{\omega_c}) + a_2(\frac{s}{\omega_c})^2 + a_3(\frac{s}{\omega_c})^3 + a_4(\frac{s}{\omega_c})^4 + a_5(\frac{s}{\omega_c})^5}$$

with zeroes at $s = \pm j\omega_1, \pm j\omega_2$

2.5.5 Design of Chebyshev Type II LPFs

Example

Design a Chebyshev Type II LPF to meet the following specifications:

- 3dB cut-off frequency at 50 Hz
- Attenuation of at least 50dB for $f \geq$ 100Hz.

Answer

1. For Chebyshev Type II LPFs, the pass-band behaviour is the same as that of a Butterworth LPF; ie. $\omega_c = \omega_{3dB}$. Therefore $|G|^2_{\omega_c} = \frac{1}{2} = \frac{1}{1+\varepsilon^2}$ which gives $\varepsilon = 1$

2. $$\text{When} \quad \omega = \omega_r, \qquad |G|^2 = \frac{1}{A^2} = \frac{1}{1 + \varepsilon^2 \frac{T_n^2(\frac{\omega_r}{\omega_c})}{T_n^2(1)}}$$

   Since $T_n^2(1) = 1$ for all n, the expression reduces to:

   $$|G|^2_{\omega_r} = \frac{1}{1 + \varepsilon^2 T_n^2(\frac{\omega_r}{\omega_c})}$$

   Therefore, from section 2.5.3,

   $$n = \frac{\cosh^{-1}(A^2 - 1)^{\frac{1}{2}}}{\cosh^{-1}(\frac{\omega_r}{\omega_c})} \qquad (\varepsilon = 1, \text{ in this case})$$

   For an attenuation of 50dB, A^2 = 10^5 and thus:

   $$n = \frac{\cosh^{-1}(10^5 - 1)^{\frac{1}{2}}}{\cosh^{-1}(2)} = 4.897$$

ie. $n = 5$ meets the overall specification.

2.5.6 Elliptic low-pass filters

Elliptic filters[3] have a magnitude response which is equi-ripple in both the pass-band and the stop-band. Elliptic filters are optimum in the sense that, for a given order and for given ripple specifications, no other filter achieves a faster transition between the pass-band and the stop-band – ie. has a narrower transition band-width.

The magnitude-squared response of a low-pass elliptic filter is of the form:

$$|G(j\omega)|^2 = \frac{1}{1+\varepsilon^2 R_n^2(\frac{\omega}{\omega_c}, L)}$$

where $R_n(\frac{\omega}{\omega_c}, L)$ is called a Chebyshev rational function and L is a parameter describing the ripple properties of $R_n(\frac{\omega}{\omega_c}, L)$.

Figure 2.10 shows the magnitude-squared response of a typical elliptic low-pass filter. The frequencies $\omega_{p,s}$ represent the edges of the pass- and stop-bands and it is noted that the cut-off frequency is given as $\omega_c = \sqrt{\omega_p \omega_s}$. The latter, once again, is not the 3dB point though.

2.5.7 Design of Elliptic Filters

In order to design an elliptic LPF with arbitrary attenuations in both pass-band and stop-band, *three* of the four parameters:

- filter order (n)
- in-band loss or ripple (ε)
- out-of-band loss or attenuation (A)
- transition ratio $R = \frac{\omega_p}{\omega_s} (< 1)$

can be chosen and the fourth parameter is uniquely defined.

The theory behind the determination of the function $R_n(\frac{\omega}{\omega_c}, L)$ involves an understanding of Jacobian elliptic functions, which are beyond the scope of this

[3] also known as Cauer, or Darlington, filters

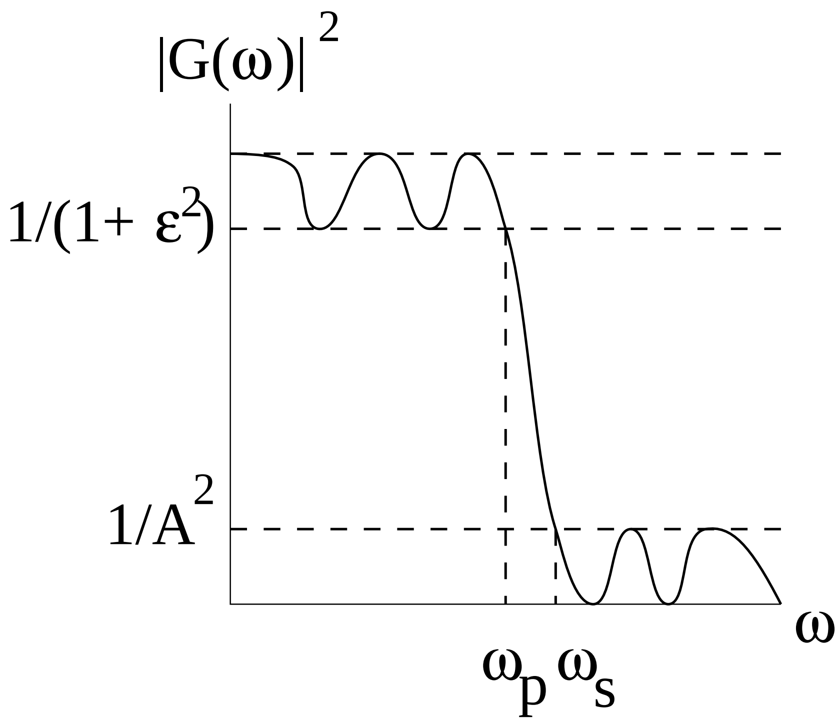

Figure 2.10: *Elliptic filter response.*

course. Hence the design of elliptic filters will not be considered in detail in this lecture course. In any case, elliptic filters are usually designed with the help of graphical procedures (see, for example, Rabiner & Gold, *Theory and Application of Digital Signal Processing*) or computer programmes (see, for example, Daniels, *Approximation methods for the design of passive, active and digital filters*).

You should remember, however, that for any given specification, *the elliptic filter will always be of lower order than any other*.