Series 3 solution of mathematical logic tutorial

Exercise 1 Solution :

1)

1. True
2. False
3. True
4. True
5. False

2)

1. True
2. False
3. False
4. False
5. False
6. True

3)

1. True
2. True
3. True
4. True
5. False
6. True

Exercise 2 Solution :

1) True, 2) False, 3) False, 4) True.

Exercise 3 Solution :

1) The truth table of $\neg(p \wedge q)$ is as follows :

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

2) The truth table of $\neg p \vee \neg q$ is as follows :

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

3) The truth table of $\neg(p \vee q)$ is as follows :

p	q	$p \vee q$	$\neg(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

4) The truth table of $\neg p \wedge \neg q$ is as follows :

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

5) The truth table of $p \vee (p \wedge q)$ is as follows :

p	q	$p \wedge q$	$p \vee (p \wedge q)$
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

6) The truth table of $p \wedge (p \vee q)$ is as follows :

p	q	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

7) The truth table of p is as follows :

p
T
F

The equivalences are :
- $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- $\neg(p \vee q) \equiv \neg p \wedge \neg q$
- $p \vee (p \wedge q) \equiv p \wedge (p \vee q) \equiv p$

Exercise 4 Solution :
1) $\neg p \equiv$ It's not cold.
2) $p \wedge q \equiv$ It's cold and it's raining.
3) $p \vee q \equiv$ It's cold or it's raining.
4) $q \vee \neg p \equiv$ It's raining or it's not cold.
5) $\neg p \wedge \neg q$ It's not cold and it's not raining.

6) $\neg\neg q$ It's not true that it's not raining (or we can say "it's raining").

Exercise 5 Solution :
1. This engine is not noisy, but it consumes a lot.
Universe of discourse : M = this engine is noisy ; C = this engine consumes a lot.
$(\neg M \wedge C)$
2. It is not true that Peter will come if Mary or John come.
Universe of discourse : P = Peter will come ; M = Mary comes ; J = John comes.
$\neg((M \vee J) \rightarrow P)$ if we interpret the sentence like this : It is not true that (Peter will come if Mary or John come) or $(M \vee J) \rightarrow \neg P$ if we interpret the sentence like this : (It is not true that Peter will come) if (Mary or John come). Different truth conditions
3. John is not only stupid, but he is also evil.
Universe of discourse : J = John is stupid ; M = John is evil.
$J \wedge M$
4. I go to the beach or to the cinema on foot or by car.
Universe of discourse : A = I go to the beach on foot ; B = I go to the cinema on foot ; C = I go to the beach by car ; D = I go to the cinema by car.
$((A \vee B) \vee (C \vee D))$
5. Peter has no brothers or sisters, but he has a cousin.
Universe of discourse : P = Peter has a brother ; Q = Peter has a sister ; R = Peter has a cousin.
$(\neg P \wedge \neg Q) \wedge R$
6.If it's raining and sunny, then there's a rainbow.
Universe of discourse : P = it's raining ; S = there is the sun ; A = there is a rainbow.
$(P \wedge S) \rightarrow A$
7. John will only go to the cinema if he has finished his homework.
Universe of discourse : C = John will go to the cinema ; D = John has finished his homework.
$(\neg D \rightarrow \neg C)$

Exercise 6 Solution :
1) First, we must introduce the propositional variables.
"A" : for " Ahmed orders a dessert ".
"L" : for " Ali orders a dessert ".
"M" : for " Mostafa orders a dessert ".

Now, we will transform the sentences into propositional logic form.

1. Sentence 1 gives : $A \longrightarrow L$,
2. Sentence 2 gives : $(L \wedge \neg M) \vee (\neg L \wedge M)$,
3. Sentence 3 gives : $A \vee M$,
4. Sentence 4 gives : $M \longrightarrow A$.

2)To really see who ordered a dessert, we make the truth table of the global formula "F_1" composed from the conjunction of all the previous statements in order to see all the

possible models.
The overall formula "F_1" $= (A \longrightarrow L) \wedge ((L \wedge \neg M) \vee (\neg L \wedge M)) \wedge (A \vee M) \wedge (M \longrightarrow A)$.
The truth table is as follows :

Interpretations	A	L	M	$A \to L$	$(L \wedge \neg M) \vee (\neg L \wedge M)$	$A \vee M$	$M \to A$	F_1
I_1	T	T	T	T	F	T	T	F
I_2	T	T	F	T	T	T	T	T
I_3	T	F	T	F	T	T	T	F
I_4	T	F	F	F	F	T	T	F
I_5	F	T	T	T	F	T	F	F
I_6	F	T	F	T	T	F	T	F
I_7	F	F	T	T	T	T	F	F
I_2	F	F	F	T	F	F	T	F

The only interpretation that makes the formula "F_1" true is the interpretation I_2 in which Ahmed and Ali order a dessert but not Mostafa. Because the truth values of propositions A and L are true but proposition M = false (in row two of the truth table).
3)If we remove one of the previous constraints, then there will always be a second model that appears and in this case we will not be able to conclude (to conclude, we must always have a single model for the formula F_1) .

Exercise 7 Solution :
1)The truth table of the formula $(p|q)$ is as follows :

p	q	$(p\|q)$
T	T	F
T	F	T
F	T	T
F	F	T

2)The truth table of the formula $((p|q)|(p|q))$ is as follows :

p	q	$(p\|q)$	$((p\|q)\|(p\|q))$
T	T	F	T
T	F	T	F
F	T	T	F
F	F	T	F

We find the truth table of $p \wedge q$
3) The connector $\neg$: $\neg p \equiv \neg(p \wedge p) \equiv (p|p)$
We can give its truth table :

p	$(p\|p)$
T	F
F	T

- The connector $\vee$: $p \vee q \equiv \neg(\neg p \wedge \neg q) \equiv (\neg p|\neg q) \equiv (p|p)|(q|q)$.

- The connector $\longrightarrow$: $p \longrightarrow q \equiv \neg p \vee q \equiv \neg(p \wedge \neg q) \equiv p|\neg q \equiv p|(q|q)$.

Exercise 8 Solution :

a) The truth table of the formula $(\neg P \wedge \neg Q) \rightarrow (\neg P \vee R)$ is as follows :

P	Q	R	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$\neg P \vee R$	$(\neg P \wedge \neg Q) \rightarrow (\neg P \vee R)$
T	T	T	F	F	F	T	T
T	T	F	F	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	F	F	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

The formula is valid.

b) The truth table of the formula $P \wedge (Q \rightarrow P) \rightarrow P$ is as follows :

P	Q	$P \rightarrow Q$	$P \wedge (Q \rightarrow P)$	$P \wedge (Q \rightarrow P) \rightarrow P$
T	T	T	T	T
T	F	T	T	T
F	T	F	F	T
F	F	T	F	T

The formula is valid.

c) The truth table of the formula $(P \vee Q) \wedge \neg P \wedge \neg Q$ is as follows :

P	Q	$P \vee Q$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$(P \vee Q) \wedge \neg P \wedge \neg Q$
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	F	F
F	F	F	T	T	T	F

The formula is unverifiable (unsatisfiable).

d) The truth table of the formula $(P \rightarrow Q) \wedge (Q \vee R) \wedge P$ is as follows :

P	Q	R	$P \rightarrow Q$	$Q \vee R$	$(Q \vee R) \wedge P$	$(P \rightarrow Q) \wedge (Q \vee R) \wedge P$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	F	T	T	F
T	F	F	F	F	F	F
F	T	T	T	T	F	F
F	T	F	T	T	F	F
F	F	T	T	T	F	F
F	F	F	T	F	F	F

The formula is verifiable (satisfiable).

e) The truth table of the formula $((P \vee Q) \rightarrow R) \leftrightarrow P$ is as follows :

P	Q	R	$P \vee Q$	$(P \vee Q) \to R$	$((P \vee Q) \to R) \leftrightarrow P$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	T	T	T
T	F	F	T	F	F
F	T	T	T	T	F
F	T	F	T	F	T
F	F	T	F	T	F
F	F	F	F	T	F

The formula is verifiable (satisfiable).

Exercise 9 Solution :

a) Put in disjunctive normal form (DNF) the formula : $(A \vee B \vee C) \wedge (C \vee \neg A)$.

$(A \vee B \vee C) \wedge (C \vee \neg A)$
$\equiv (A \wedge (C \vee \neg A)) \vee (B \wedge (C \vee \neg A)) \vee (C \wedge (C \vee \neg A))$ The distributivity
$\equiv ((A \wedge C) \vee (A \wedge \neg A)) \vee ((B \wedge C) \vee (B \wedge \neg A)) \vee ((C \wedge C) \vee (C \wedge \neg A))$ The distributivity
$\equiv (A \wedge C) \vee ((B \wedge C) \vee (B \wedge \neg A)) \vee (C \vee (C \wedge \neg A))$ We removed $(A \wedge \neg A) =$ False according to the rule of unsatisfiability.
$\equiv (A \wedge C) \vee ((B \wedge C) \vee (B \wedge \neg A)) \vee C$ We replaced $(C \vee (C \wedge \neg A))$ by C depending on absorption.
$\equiv (A \wedge C) \vee (B \wedge C) \vee (B \wedge \neg A) \vee C$

b) Put in disjunctive normal form (DNF) the formula : $(A \vee B) \wedge (C \vee D)$.

$(A \vee B) \wedge (C \vee D)$
$\equiv (A \wedge (C \vee D)) \vee (B \wedge (C \vee D))$ The distributivity
$\equiv ((A \wedge C) \vee (A \wedge D)) \vee ((B \wedge C) \vee (B \wedge D))$ The distributivity
$\equiv (A \wedge C) \vee (A \wedge D) \vee (B \wedge C) \vee (B \wedge D)$

c) Put in disjunctive normal form (DNF) the formula : $\neg((A \vee B) \to C)$.

$\neg((A \vee B) \to C)$
$\equiv \neg(\neg(A \vee B) \vee C)$ The transformation of implication into disjunction.
$\equiv (A \vee B) \wedge \neg C$ We distributed the $\neg$.
$\equiv \neg C \wedge (A \vee B)$ The commutativity of $\wedge$.
$\equiv (\neg C \wedge A) \vee (\neg C \wedge B)$ The distributivity.

Exercise 10 Solution :

a) Put in conjunctive normal form (CNF) the formula : $(A \vee B) \to (C \wedge D)$.

$(A \vee B) \to (C \wedge D)$
$\equiv \neg(A \vee B) \vee (C \wedge D)$ The transformation of implication into disjunction.
$\equiv (\neg A \wedge \neg B) \vee (C \wedge D)$ Morgan's law.
$\equiv (\neg A \vee (C \wedge D)) \wedge (\neg B \vee (C \wedge D))$ The distributivity.
$\equiv ((\neg A \vee C) \wedge (\neg A \vee D)) \wedge ((\neg B \vee C) \wedge (\neg B \vee D))$ The distributivity.
$\equiv (\neg A \vee C) \wedge (\neg A \vee D) \wedge (\neg B \vee C) \wedge (\neg B \vee D)$

b) Put in conjunctive normal form (CNF) the formula : $(A \lor (\neg B \land (C \lor (\neg D \land E))))$.

$(A \lor (\neg B \land (C \lor (\neg D \land E))))$
$\equiv (A \lor (\neg B \land ((C \lor \neg D) \land (C \lor E))))$ The distributivity.
$\equiv ((A \lor \neg B) \land (A \lor ((C \lor \neg D) \land (C \lor E))))$ The distributivity.
$\equiv ((A \lor \neg B) \land ((A \lor C \lor \neg D) \land (A \lor C \lor E)))$ The distributivity.
$\equiv (A \lor \neg B) \land (A \lor C \lor \neg D) \land (A \lor C \lor E)$

c) Put in conjunctive normal form (CNF) the formula : $A \leftrightarrow (B \land \neg C)$.

$A \leftrightarrow (B \land \neg C)$
$\equiv (A \rightarrow (B \land \neg C)) \land ((B \land \neg C) \rightarrow A)$ The transformation of $\leftrightarrow$ in double $\rightarrow$.
$\equiv (\neg A \lor (B \land \neg C)) \land (\neg(B \land \neg C) \lor A)$ The transformation of implication into disjunction.
$\equiv (\neg A \lor (B \land \neg C)) \land ((\neg B \lor C) \lor A)$ Morgan's law.
$\equiv ((\neg A \lor B) \land (\neg A \lor \neg C)) \land ((\neg B \lor C) \lor A)$ The distributivit.
$\equiv (\neg A \lor B) \land (\neg A \lor \neg C) \land (\neg B \lor C \lor A)$.

Exercise 11 Solution :
a) $\vdash A \leftrightarrow A$
1$\vdash A \rightarrow (A \rightarrow A)$ sch 1a, replace B with A
2$\vdash (A \rightarrow (A \rightarrow A)) \rightarrow ((A \rightarrow ((A \rightarrow A) \rightarrow A)) \rightarrow (A \rightarrow A))$ she 1b, replace B with $A \rightarrow A$ and C with A
3$\vdash (A \rightarrow ((A \rightarrow A) \rightarrow A)) \rightarrow (A \rightarrow A)$ mp 1,2
4$\vdash A \rightarrow ((A \rightarrow A) \rightarrow A)$ sch 1a, replace B with $A \rightarrow A$
5$\vdash A \rightarrow A$ mp 3,4
6$\vdash (A \rightarrow A) \rightarrow ((A \rightarrow A) \rightarrow (A \leftrightarrow A))$ sch 9, replace B with A
7$\vdash (A \rightarrow A) \rightarrow (A \leftrightarrow A)$ mp 5,6
8$\vdash A \leftrightarrow A$ mp 5,7

b) $\vdash \neg B \rightarrow (B \rightarrow A)$
1$\vdash (\neg B \rightarrow (\neg A \rightarrow \neg B)) \rightarrow (((\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)) \rightarrow (\neg B \rightarrow (B \rightarrow A))$ sch 1d, replace A with $\neg B$, B with $(\neg A \rightarrow \neg B)$ and C with $B \rightarrow A$
2$\vdash \neg B \rightarrow (\neg A \rightarrow \neg B)$ sch 1a, replace A with $\neg B$ and B with $\neg A$
3$\vdash ((\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)) \rightarrow (\neg B \rightarrow (B \rightarrow A))$ mp 1,2
4$\vdash (\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$ sch 8
5$\vdash \neg B \rightarrow (B \rightarrow A)$ mp 3,4

Exercise 12 Solution :
a) $A \rightarrow (B \rightarrow C), A \land B \vdash C$
1$\vdash A \rightarrow (B \rightarrow C)$ hyp 1
2$\vdash A \land B$ hyp 2
3$\vdash A \land B \rightarrow A$ sch 3a
4$\vdash A$ mp 2,3
5$\vdash B \rightarrow C$ mp 4,1

6$\vdash A \wedge B \to B$ sch 3b
7$\vdash B$ mp 2,6
8$\vdash C$ mp 7,5

b) $A \to (B \to C), B \vdash A \to C$
1$\vdash A \to (B \to C)$ hyp 1
2$\vdash B$ hyp 2
3$\vdash (A \to B) \to ((A \to (B \to C)) \to (A \to C))$ sch 1b
4$\vdash B \to (A \to B)$ sch 1a, replace A with B and B with A
5$\vdash (A \to B)$ mp 2,4
6$\vdash (A \to (B \to C)) \to (A \to C)$ mp 5,3
7$\vdash A \to C$ mp 1,6

c) $A, B \wedge C, A \wedge C \to E \vdash E$
1$\vdash A$ hyp 1
2$\vdash B \wedge C$ hyp 2
3$\vdash A \wedge C \to E$ hyp 3
4$\vdash A \to (C \to A \wedge C)$ sch 2, replace B with C
5$\vdash C \to A \wedge C$ mp 1,4
6$\vdash B \wedge C \to C$ sch 3b, replace A with B and B with C
7$\vdash C$ mp 2,6
8$\vdash A \wedge C$ mp 7,5
9$\vdash E$ mp 8,3

d) $E, E \to (A \wedge D), D \vee F \to G \vdash G$
1$\vdash E$ hyp 1
2$\vdash E \to (A \wedge D)$ hyp 2
3$\vdash D \vee F \to G$ hyp 3
4$\vdash A \wedge D$ mp 1,3
5$\vdash A \wedge D \to D$ sch 3b, replace B with D
6$\vdash D$ mp 4,5
7$\vdash D \to d \vee F$ sch 4a, replace A with D and B with F
8$\vdash D \vee F$ mp 6,7
9$\vdash G$ mp 8,3

e) $\neg A \to C, (\neg C \to A) \to ((C \to A) \to A) \vdash (\neg A \to \neg C) \to A$
1 $\vdash \neg A \to C$ Hyp1
2 $\vdash (\neg C \to A) \to ((C \to A) \to A)$ Hyp2
3 $\vdash (\neg A \to C) \to (\neg C \to A)$ sch 8 replace B with $\neg C$
4 $\vdash \neg C \to A$ mp 1,3
5 $\vdash (C \to A) \to A$ mp 2,4
6 $\vdash ((\neg A \to \neg C) \to (C \to A)) \to ((C \to A) \to A) \to ((\neg A \to \neg C) \to A))$ sch 1d replace A with $\neg A \to \neg C$, B with $C \to A$ and C with A
7 $\vdash (\neg A \to \neg C) \to (C \to A)$ sch 8 replace B with C
8 $\vdash ((C \to A) \to A) \to ((\neg A \to \neg C) \to A))$ mp 6,7
9 $\vdash (\neg A \to \neg C) \to A$ mp 5,8