Series 4 solution of first order logic tutorial

Exercise 1 Solution :
First, we need to specify the vocabulary used (the domain of discourse). In our case, we will use the following vocabulary :
Constants :
log : It is the constant representing logic.
Variables :
x and y.
The unary predicates :
Stud(x) : means that "x" is a student.
Module(y) : means that "y" is a module.
The binary predicates :
like(x,y) : means that "x" likes "y".
gnote(x,y) :means that "x" got a good grade in "y".
Better(x,y) : means that "x" is better than "y".
Now we will model the sentences :
1- $\forall x(Stud(x) \longrightarrow like(x, log))$
2- $\forall x \exists y((Stud(x) \wedge Module(y)) \longrightarrow \neg like(x, y))$
3- $\forall x \forall y((Stud(x) \wedge gnote(x, log) \wedge Stud(y)) \longrightarrow Better(x, y))$

Exercise 2 Solution :
1- The predicates are :
man(x) and mortal(x).

2- The sentences in predicate logic are :
a- $\forall x(man(x) \longrightarrow mortal(x))$.
b- mortal(Socrate).
c- man(Socrate).

3-Yes, we can deduce statement "b" from "a" and "c".
Proof :
Intuitively, "Socrates" is an instance of "x", and since "Socrates" is a man (man(Socrates) = true) and $\forall x(man(x) \longrightarrow mortal(x)) = true$ so "Socrate" is mortal that is, mortal(Socrates) = true.
We can also prove this using proof theory, that is, we need to prove that :
$\forall x(man(x) \longrightarrow mortal(x)), man(Socrate) \vdash mortal(Socrate)$.
The proof is like this :
1- $\vdash \forall x(man(x) \longrightarrow mortal(x))$ hyp1
2- $\vdash man(Socrate)$ hyp2
3- $\vdash \forall x(man(x) \longrightarrow mortal(x)) \longrightarrow (man(Socrate) \longrightarrow mortal(Socrate))$ sch$\forall$
4-$\vdash man(Socrate) \longrightarrow mortal(Socrate)$ mp 1,3
5- $\vdash mortal(Socrate)$ mp 2,4

Exercise 3 Solution :
1- $\forall x(influenza(x) \longrightarrow take(x, Tamiflu))$
2- $\forall x((fever(x) \wedge cough(x)) \longrightarrow influenza(x))$
3- $\forall x \forall t((temp(x,t) \wedge sup(t,38)) \longrightarrow fever(x))$
4- $cough(Mohamed) \wedge \exists t(temp(Mohamed,t) \wedge sup(t,38))$
5- $take(Mohamed, Tamiflu)$

Exercise 4 Solution :
To answer this question, we need to construct the syntax trees of each formula.
After numbering the variables, we find :
$F_1 : \forall x_4 \exists y_3(\forall y_1 P(x_4, y_1) \rightarrow \exists x_2 Q(x_2, y_3)) \Rightarrow F_1 : \forall 4 \exists 3(\forall 1 P(4,1) \rightarrow \exists 2 Q(2,3))$
$F_2 : \forall v_4 \exists z_3(\forall u_1 P(z_3, u_1) \rightarrow \exists u_2 Q(u_2, v_4)) \Rightarrow F_2 : \forall 4 \exists 3(\forall 1 P(3,1) \rightarrow \exists 2 Q(2,4))$
$F_4 : \forall z_4 \exists x_3(\forall x_1 P(z_4, x_1) \rightarrow \exists z_2 Q(z_2, x_3)) \Rightarrow F_4 : \forall 4 \exists 3(\forall 1 P(4,1) \rightarrow \exists 2 Q(2,3))$
According to results, we observe that F_1 and F_3 are congruent.

Exercise 5 Solution :
-1 False, **2-** False, **3-** True, **4-** False, **5-** True, **6-** True, **7-** True, **8-**False.
For justification, for example, for the first one : Q(c1) = True so $\neg Q(c1)$ = False, Q(c2) = True so $\neg Q(c2)$ = False and Q(c3) = True so $\neg Q(c3)$ = False. $I(\forall x \neg Q(x))$ = $I(\neg Q(c1) \wedge \neg Q(c2) \wedge \neg Q(c3)) = F \wedge F \wedge F =$ F. We do the same for the other formulas.

Exercise 6 Solution :
a)The truth table of the first formula $\forall x(P(x) \longrightarrow \exists x Q(x))$ knowing that the domain D $= \{1, 2\}$.

— We will represent the predicate P(x) by the function f(x),
— The predicate Q(x) will be represented by the function g(x).

The truth table of P(x) is as follows :

x P(x)	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$
1	T	T	F	F
2	T	F	T	F

The truth table of Q(x) is as follows :

x Q(x)	$g_1(x)$	$g_2(x)$	$g_3(x)$	$g_4(x)$
1	T	T	F	F
2	T	F	T	F

Since there are no free variables in the formula, the overall truth table is as follows :

Interpretation	P(x)	Q(x)	Formula
I_1	f_1	g_1	T
I_2	f_1	g_2	T
I_3	f_1	g_3	T
I_4	f_1	g_4	F
I_5	f_2	g_1	T
I_6	f_2	g_2	T
I_7	f_2	g_3	T
I_8	f_2	g_4	F
I_9	f_3	g_1	T
I_{10}	f_3	g_2	T
I_{11}	f_3	g_3	T
I_{12}	f_3	g_4	F
I_{13}	f_4	g_1	T
I_{14}	f_4	g_2	T
I_{15}	f_4	g_3	T
I_{16}	f_4	g_4	T

To provide the truth values of each interpretation, follow the following method :

I_1 (T) $\forall x\{and^{x=1(T)f_1(1)(T)\rightarrow\exists x(T)\{or^{x=1g_1(1)(T)}_{x=2g_1(2)(T)}}_{x=2(T)f_1(2)(T)\rightarrow\exists x(T)\{or^{x=1g_1(1)(T)}_{x=2g_1(2)(T)}}$

I_2 (T) $\forall x\{and^{x=1(T)f_1(1)(T)\rightarrow\exists x(T)\{or^{x=1g_2(1)(t)}_{x=2g_2(2)(F)}}_{x=2(T)f_1(2)(T)\rightarrow\exists x(T)\{or^{x=1g_2(1)(T)}_{x=2g_2(2)(F)}}$

I_4 (F) $\forall x\{and^{x=1(F)f_1(1)(T)\rightarrow\exists x(F)\{or^{x=1g_4(1)(F)}_{x=2g_4(2)(F)}}_{x=2(F)f_1(2)(T)\rightarrow\exists x(F)\{or^{x=1g_4(1)(F)}_{x=2g_4(2)(F)}}$

I_8 (F) $\forall x\{and^{x=1(F)f_2(1)(T)\rightarrow\exists x(F)\{or^{x=1g_4(1)(F)}_{x=2g_4(2)(F)}}_{x=2(T)f_2(2)(F)\rightarrow\exists x(F)\{or^{x=1g_4(1)(F)}_{x=2g_4(2)(F)}}$

b) The truth table of the second formula $\forall x(P(x,y) \wedge \exists x P(x))$.
In this case, we have two predicates and one free variable y.

— We will represent the predicate P(x) by the function f(x),
— The predicate P(x, y) will be represented by the function g(x).

The truth table of P(x) is as follows :

x P(x)	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$
1	T	T	F	F
2	T	F	T	F

The truth table of P(x,y) is as follows :

x	y	$g_1(x,y)$	$g_2(x,y)$	$g_3(x,y)$	$g_4(x,y)$	$g_5(x,y)$	$g_6(x,y)$	$g_7(x,y)$	$g_8(x,y)$	$g_9(x,y)$
1	1	T	T	T	T	T	T	T	T	F
1	2	T	T	T	T	F	F	F	F	T
2	1	T	T	F	F	T	T	F	F	T
2	2	T	F	T	F	T	F	T	F	T

$g_{10}(x,y)$	$g_{11}(x,y)$	$g_{12}(x,y)$	$g_{13}(x,y)$	$g_{14}(x,y)$	$g_{15}(x,y)$	$g_{16}(x,y)$
F	F	F	F	F	F	F
T	T	T	F	F	F	F
T	F	F	T	T	F	F
F	T	F	T	F	T	F

The overall truth table is as follows :

Interpretation	P(x)	P(x,y)	y	Formula
I_1	f_1	g_1	1	T
I_2	f_1	g_1	2	.
I_3	f_1	g_2	1	.
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
.	f_1	g_{16}	1	.
.	f_1	g_{16}	2	.
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
I_{98}	f_4	g_1	2	F
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

To provide the truth values of each interpretation, follow the following method :

$$I_1\ (\mathrm{T})\ \forall x\{and^{x=1(T)g_1(1,1)(T)\wedge\exists x(T)\{or^{x=1f_1(1)(T)}_{x=2f_1(2)(T)}}_{x=2(T)g_1(2,1)(T)\wedge\exists x(T)\{or^{x=1f_1(1)(T)}_{x=2f_1(2)(T)}}$$

$$I_{98}\ (\mathrm{F})\ \forall x\{and^{x=1(F)g_1(1,2)(T)\wedge\exists x(F)\{or^{x=1f_4(1)(F)}_{x=2f_4(2)(F)}}_{x=2(F)g_1(2,2)(T)\wedge\exists x(F)\{or^{x=1f_4(1)(F)}_{x=2f_4(2)(F)}}$$

Exercise 7 Solution :

a)

1) $\forall x\forall yA(x,y)\vdash A(x,y)$

1$\vdash \forall x\forall yA(x,y)$ hyp1

2$\vdash \forall x\forall yA(x,y)\longrightarrow\forall yA(x,y)$ sch$\forall$

3$\vdash \forall yA(x,y)$ mp 1,2

4$\vdash \forall yA(x,y)\longrightarrow A(x,y)$ sch$\forall$

5$\vdash A(x,y)$ mp 3,4

2) $\forall x(P(x)\longrightarrow Q(x)),\forall xP(x)\vdash\forall xQ(x)$

1$\vdash \forall x(P(x)\longrightarrow Q(x))$ hyp1

2$\vdash \forall x P(x)$ hyp2
3$\vdash \forall x(P(x) \longrightarrow Q(x)) \longrightarrow (P(x) \longrightarrow Q(x))$ sch$\forall$
4$\vdash P(x) \longrightarrow Q(x)$ mp 1,3
5$\vdash \forall x P(x) \longrightarrow P(x)$ sch$\forall$
6$\vdash P(x)$ mp 2,5
7$\vdash Q(x)$ mp 4,6
8$\vdash Q(x) \longrightarrow (\forall x P(x) \longrightarrow Q(x))$ sch 1a
9$\vdash \forall x P(x) \longrightarrow Q(x)$ mp 7,8
10$\vdash \forall x P(x) \longrightarrow \forall x Q(x)$ rule $\forall$ from 9
11$\vdash \forall x Q(x)$ mp 2,10

3) $P(a), \forall x(P(x) \longrightarrow Q(x)) \vdash Q(a)$
1$\vdash P(a)$ hyp1
2$\vdash \forall x(P(x) \longrightarrow Q(x))$ hyp2
3$\vdash \forall x(P(x) \longrightarrow Q(x)) \longrightarrow (P(a) \longrightarrow Q(a))$ sch$\forall$
4$\vdash P(a) \longrightarrow Q(a)$ mp 2,3
5$\vdash Q(a)$ mp 1,4

4) $\forall x S(x) \wedge \forall x R(x) \vdash \exists x(S(x) \wedge R(x))$
1 $\vdash \forall x S(x) \wedge \forall x R(x)$ Hyp1
2 $\vdash \forall x S(x) \wedge \forall x R(x) \rightarrow \forall x S(x)$ sch 3a we replaced A with $\forall x S(x)$ and B with $\forall x R(x)$
3 $\vdash \forall x S(x)$ mp 1,2
4 $\vdash \forall x S(x) \wedge \forall x R(x) \rightarrow \forall x R(x)$ sch 3bwe replaced A with $\forall x S(x)$ and B with $\forall x R(x)$
5 $\vdash \forall x R(x)$ mp 1,4
6 $\vdash \forall x S(x) \rightarrow S(x)$ sch $\forall$
7 $\vdash S(x)$ mp 3,6
8 $\vdash \forall x R(x) \rightarrow R(x)$ sch $\forall$
9 $\vdash R(x)$ mp 5,8
10 $\vdash S(x) \rightarrow (R(x) \rightarrow S(x) \wedge R(x))$ sch 2 we replaced A with $S(x)$ and B with $R(x)$
11 $\vdash R(x) \rightarrow S(x) \wedge R(x)$ mp 7,10
12 $\vdash S(x) \wedge R(x)$ mp 9,11
13 $\vdash (S(x) \wedge R(x)) \rightarrow \exists x(S(x) \wedge R(x))$ sch $\exists$
14 $\vdash \exists x(S(x) \wedge R(x))$ mp 12,13

b)
$\vdash \forall x P(x) \longrightarrow \exists x P(x)$
1$\vdash \forall x P(x) \longrightarrow P(x)$ sch$\forall$
2$\vdash (\forall x P(x) \rightarrow P(x)) \rightarrow ((\forall x P(x) \rightarrow (P(x) \rightarrow \exists x P(x))) \rightarrow (\forall x P(x) \rightarrow \exists x P(x)))$ sch 1b, we replaced A with $\forall x P(x)$, B with P(x) and C with $\exists x P(x)$
3$\vdash (\forall x P(x) \rightarrow (P(x) \rightarrow \exists x P(x))) \rightarrow (\forall x P(x) \rightarrow \exists x P(x))$ mp 1,2
4$\vdash P(x) \longrightarrow \exists x P(x)$ sch$\exists$
5$\vdash (P(x) \rightarrow \exists x P(x)) \rightarrow (\forall x P(x) \rightarrow (P(x) \rightarrow \exists x P(x)))$ sch 1a we replaced A with $(P(x) \rightarrow \exists x P(x))$ and B par $\forall x P(x)$
6$\vdash (\forall x P(x) \rightarrow (P(x) \rightarrow \exists x P(x)))$ mp 4,5
7$\vdash \forall x P(x) \longrightarrow \exists x P(x)$ mp 3,6