SOLUTION OF Series 1

<u>Exercise 1:</u>

Let's determine (if they exist): the upper bounds, the lower bounds, the supremum, the infimum, the maximum, and the minimum of the following sets:

1. (a) For the set $[-1;3]$, We have:
   - The upper bounds of $[-1;3]$ are: $[3;+\infty[$.
   - The lower bounds of $[-1;3]$ are: $]-\infty;-1]$.
   - The supremum of $[-1;3]$ is: $\sup([-1;3]) = 3$.
   - The infimum of $[-1;3]$ is: $\inf([-1;3]) = -1$.
   - The greatest element of $[-1;3]$ is: $\max([-1;3]) = 3$.
   - The least element of $[-1;3]$ is: $\min([-1;3]) = -1$.

   (b) For the set $]0;1[$, We have:
   - The upper bounds of $]0;1[$ are: $[1;+\infty[$.
   - The lower bounds of $]0;1[$ are: $]-\infty;0]$.
   - The supremum of $]0;1[$ is: $\sup(]0;1[) = 1$.
   - The infimum of $]0;1[$ is: $\inf(]0;1[) = 0$.
   - The greatest element of $]0;1[$: does not exist.
   - The least element of $]0;1[$: does not exist.

   (c) For the set $[-1;3]\cap\mathbb{Z}$, We have: $[-1;3]\cap\mathbb{Z} = \{-1,0,1,2,3\}$; therefore:
   - The upper bounds of $[-1;3]\cap\mathbb{Z}$ are: $[3;+\infty[$.
   - The lower bounds of $[-1;3]\cap\mathbb{Z}$ are: $]-\infty;-1]$.
   - The supremum of $[-1;3]\cap\mathbb{Z}$ is: $\sup([-1;3]\cap\mathbb{Z}) = 3$.
   - The infimum of $[-1;3]\cap\mathbb{Z}$ is: $\inf([-1;3]\cap\mathbb{Z}) = -1$.
   - The greatest element of $[-1;3]\cap\mathbb{Z}$ is: $\max([-1;3]\cap\mathbb{Z}) = 3$.
   - The least element of $[-1;3]\cap\mathbb{Z}$ is: $\min([-1;3]\cap\mathbb{Z}) = -1$.

   (d) For the set $\mathbb{N}$, We have:
   - The upper bounds of $\mathbb{N}$ do not exist.
   - The lower bounds of $\mathbb{N}$ are: $]-\infty;0]$.
   - The supremum of $\mathbb{N}$ does not exist.
   - The infimum of $\mathbb{N}$ is: $\inf(\mathbb{N}) = 0$.
   - The greatest element of $\mathbb{N}$ does not exist.
   - The least element of $\mathbb{N}$ is: $\min(\mathbb{N}) = 0$.

2. For the set $A = \left\{\frac{1}{n} + \frac{1}{n^2}\ ;\ n \in \mathbb{N}^*\right\}$.

   We have: $\forall n \in \mathbb{N}^*$: $\frac{1}{n}$ and $\frac{1}{n^2}$ are decreasing for $n \geq 1$, so we have:

$$\begin{cases} n \geq 1 \Rightarrow \frac{1}{n} \leq 1 \\ n \geq 1 \Rightarrow \frac{1}{n^2} \leq 1 \end{cases} \Rightarrow \begin{cases} 0 < \frac{1}{n} \leq 1 \\ 0 < \frac{1}{n^2} \leq 1 \end{cases} \Rightarrow 0 < \frac{1}{n} + \frac{1}{n^2} \leq 2$$

   Thus,

- The upper bounds of A are: $[2;+\infty[$.
- The lower bounds of A are: $]-\infty;0]$.
- The supremum of A is: $\sup(A)=2$.
- The infimum of A is: $\inf(A)=0$.
- The greatest element of A is: $\max(A)=2$.
- The least element of A does not exist.

3. For the set $B=\left\{\frac{1}{2x+1}\ ;\ x\in[0,1]\right\}$.

   We have:

$$\begin{aligned} x\in[0;1] &\Rightarrow 0\leq x\leq 1\\ &\Rightarrow 0\leq 2x\leq 2\\ &\Rightarrow 1\leq 2x+1\leq 3\\ &\Rightarrow \frac{1}{3}\leq\frac{1}{2x+1}\leq 1. \end{aligned}$$

   Thus,

   - The upper bounds of B are: $[1;+\infty[$.
   - The lower bounds of B are: $]-\infty,\frac{1}{3}]$.
   - The supremum of B is: $\sup(B)=1$.
   - The infimum of B is: $\inf(B)=\frac{1}{3}$.
   - The greatest element of B is: $\max(B)=1$.
   - The least element of B is: $\min(B)=\frac{1}{3}$.

Exercise 2:

1. Let's write the following expressions without absolute value:

   (a) $f(x)=3+|x-1|$
   We have:

$$|x-1|=\begin{cases} x-1 & \text{if } x\in[1;+\infty[\\ -x+1 & \text{if } x\in]-\infty;1[\end{cases}$$

   Thus,

$$f(x)=\begin{cases} x+2 & \text{if } x\in[1;+\infty[\\ -x+4 & \text{if } x\in]-\infty;1[\end{cases}$$

   (b) $g(x)=|x-2|+|2x+3|$
   We have:

$$|x-2|=\begin{cases} x-2 & \text{if } x\in[2;+\infty[\\ -x+2 & \text{if } x\in]-\infty;2[\end{cases}$$

and

$$|2x+3| = \begin{cases} 2x+3 & \text{if } x \in \left[-\frac{3}{2};+\infty\right[\\ -2x-3 & \text{if } x \in]-\infty;-\frac{3}{2}[\end{cases}$$

Thus,

$$g(x) = \begin{cases} 3x+1 & \text{if } x \in [2;+\infty[\\ -3x-1 & \text{if } x \in]-\infty;-\frac{3}{2}[\\ x+5 & \text{if } x \in \left[-\frac{3}{2};2\right[\end{cases}$$

2. a.

$$\begin{aligned} |x-2| &= 3 \Leftrightarrow x-2=3 \quad \text{or } x-2=-3 \\ &\Leftrightarrow \ x=5 \qquad\qquad \text{or } x=-1 \\ &\Leftrightarrow \ S=\{-1,(\} \end{aligned}$$

b.

$$\begin{aligned} |2x-6| &= |x+1| \Leftrightarrow 2x-6=x+1 \ \text{ or } \ 2x-6=-x-1 \\ &\Leftrightarrow \ 2x-x=7 \qquad\qquad \text{or } \ 2\ x+x=5 \\ &\Leftrightarrow \ S=\left\{\frac{5}{3},7\right\} \end{aligned}$$

c.

$$\begin{aligned} 2 &\leq \left|x^2-1\right| \ \ldots\ldots\ldots\ldots\ldots\ldots (*) \\ if\ \ x &\in \ ]-\infty,-1]\cup[1,+\infty[=A \quad \text{then} \quad x^2-1\geq 0 \Leftrightarrow \left|x^2-1\right| = x^2-1 \\ (*) &\Leftrightarrow \quad 2\leq x^2-1 \quad \Leftrightarrow\ 3\leq x^2 \ \Leftrightarrow\ \sqrt{3}\leq\sqrt{x^2} \ \Leftrightarrow\ |x|\geq\sqrt{3} \\ &\Leftrightarrow \ x\in\left]-\infty,-\sqrt{3}\right]\cup\left[\sqrt{3},+\infty\right[=B \\ we\ have\quad B\cap A &= B \quad (B\subset A) \\ if\ x &\in \ ]-1,1[\Leftrightarrow x^2-1<0\Leftrightarrow\left|x^2-1\right|=-x^2+1 \\ (*) &\Leftrightarrow \quad 2\leq -x^2+1 \ \Leftrightarrow x^2\leq 1-2\Leftrightarrow x^2\leq -1 \text{ impossible} \\ Then \quad S &= B \end{aligned}$$

0.1 Exercise 3:

Let us demonstrate that for all $x, y \in \mathbb{R}$:

$$\max(x,y) = \frac{x+y+|x-y|}{2}, \quad \text{and} \quad \min(x,y) = \frac{x+y-|x-y|}{2}.$$

1. If $x \geq y$, we have: $\max(x,y) = x$ and $\min(x,y) = y$, and

$$\frac{x+y+|x-y|}{2} = \frac{x+y+(x-y)}{2} = \frac{2x}{2} = x,$$

and

$$\frac{x+y-|x-y|}{2}=\frac{x+y-(x-y)}{2}=\frac{2y}{2}=y.$$

If $x \leq y$, we have: $\max(x,y)=y \quad \text{and} \quad \min(x,y)=x$, and

$$\frac{x+y+|x-y|}{2}=\frac{x+y-(x-y)}{2}=\frac{2y}{2}=y,$$

and

$$\frac{x+y-|x-y|}{2}=\frac{x+y+(x-y)}{2}=\frac{2x}{2}=x.$$

Thus, for all $x,y\in\mathbb{R}$:

$$\max(x,y)=\frac{x+y+|x-y|}{2}$$

and

$$\min(x,y)=\frac{x+y-|x-y|}{2}.$$

0.2 Exercise 4:

1.

$$E\left(\frac{x-1}{2}\right)=-2$$

$$\begin{aligned}
E(x) &= k, k\in\mathbb{Z} \Leftrightarrow k\leq x<k+1\\
0 &\leq \frac{x-1}{2}<-2+1 \Leftrightarrow -4\leq x-1<-2\\
&\Leftrightarrow -3\leq x<-1 \Leftrightarrow S=[-3,-1[
\end{aligned}$$

2.

$$\begin{aligned}
E\left(2x\right) &= x-1 \Leftrightarrow x-1\leq 2x<x \quad \text{and} \;\; x-1\in\mathbb{Z}\\
&\Leftrightarrow x-1\leq 2x \quad \text{and } 2x<x \quad \text{and} \;\; x-1\in\mathbb{Z}\\
&\Leftrightarrow -1\leq x \quad \text{and } x<0 \quad \text{and} \;\; x\in\mathbb{Z}\\
&\Leftrightarrow -1\leq x \;\; <0 \text{ and } \;\; x\in\mathbb{Z}\\
&\Leftrightarrow x=-1
\end{aligned}$$

3.

$$E(x)+|x-1|=x$$

- The frst case: $x-1\geq 0 \Leftrightarrow x\geq 1$

$$\begin{aligned}
E(x)+|x-1| &= x \Leftrightarrow E(x)+x-1=x\\
&\Leftrightarrow E(x)=1\in\mathbb{Z}\Leftrightarrow 1\leq x<2 \Leftrightarrow x\in[1,2[
\end{aligned}$$

- The second case:

$$\begin{aligned}
E(x)+|x-1| &= x \Leftrightarrow E(x)-x+1=x\\
&\Leftrightarrow E(x)=2x-1\\
&\Leftrightarrow 2x-1 \le x < 2x-1+1 \quad \text{and } 2x-1\in\mathbb{Z}\\
&\Leftrightarrow -1 \le x-2x < 0 \quad \text{and } 2x\in\mathbb{Z}\\
&\Leftrightarrow -1 \le -x < 0 \quad \text{and } 2x\in\mathbb{Z}\\
&\Leftrightarrow 0 < x \le 1 \quad \text{and } 2x\in\mathbb{Z}\\
&\Leftrightarrow x=1 \text{ and } x=\frac{1}{2} \text{ because } x=\frac{1}{2},\ 2x\in\mathbb{Z}, \left(2\times\frac{1}{2}\in\mathbb{Z}\right)\\
&\Leftrightarrow S=[1,2[\cap\left\{\frac{1}{2}\right\}
\end{aligned}$$

0.3 <u>Exercise 5:</u>

$x \ge \frac{1}{2}, y \le 1, x-y=3$

1.

$$\begin{aligned}
E &= \sqrt{(2x-1)^2}+\sqrt{(2y-2)^2}\\
&= |2x-1|+|2y-2|
\end{aligned}$$

$$\begin{aligned}
|2x-1| &= 2x-1 \quad \text{and} \quad |2y-2|=-2y+2\\
x &\ge \frac{1}{2} \Leftrightarrow 2x\ge 1 \Leftrightarrow 2x-1\ge 0\\
y &\le 1 \Leftrightarrow 2y \le 2 \Leftrightarrow 2y-2\le 0
\end{aligned}$$

Thus

$$\begin{aligned}
E &= 2x-1-2y+2\\
&= 2x-2y+1\\
&= 2(x-y)+1\\
&= 2\times 3+1=7
\end{aligned}$$

2. Show that $\frac{1}{2} \le x \le 4$ and -$\frac{5}{2} \le y \le 1$

   we have

$$\begin{aligned}
x-y &= 3 \Leftrightarrow y=x-3\\
\text{with } y &\le 1 \quad \text{then } x-3\le 1 \ \Leftrightarrow \frac{1}{2}\le x\le 4
\end{aligned}$$

and

$$\begin{aligned}
x - y &= 3 \Leftrightarrow x = y + 3 \\
with\ \frac{1}{2} &\leq x \quad \text{then } \frac{1}{2} \leq y + 3 \\
&\Leftrightarrow \frac{1}{2} - 3 \leq y \\
&\Leftrightarrow \frac{5}{2} \leq y \leq 1
\end{aligned}$$

3.

$$F = |x + y - 5| + |x + y + 2|$$

we have

$$\begin{aligned}
\frac{1}{2} &\leq x \leq 4 \quad and \quad -\frac{5}{2} \leq y \leq 1 \\
&\Leftrightarrow \frac{1}{2} - \frac{5}{2} \leq x + y \leq 4 + 1 \Leftrightarrow -2 \leq x + y \leq 5 \\
&\Leftrightarrow -2 - 5 \leq x + y - 5 \leq 5 - 5 \\
&\Leftrightarrow -7 \leq x + y - 5 \leq 0 \text{ and we have } 0 \leq x + y + 2 \leq 7
\end{aligned}$$

then

$$\begin{cases} |x + y - 5| = -x - y + 5 \\ |x + y + 2| = x + y + 2 \end{cases}$$

then

$$F = -x - y + 5x + y + 2 = 7$$