Solution of Series 2

Exercise 1 : 1. Study of the convergence of sequences :

a. $U_n = \sqrt{n^2+n+1} - \sqrt{n} = \sqrt{n\left(n+1+\frac{1}{n}\right)} - \sqrt{n} = \sqrt{n}\left(\sqrt{n+1+\frac{1}{n}} - 1\right)$

So, $\lim_{n\to+\infty} U_n = +\infty \implies (U_n)$ is divergent.

b. $U_n = \frac{3^n + (-3)^n}{3^n} = \frac{3^n}{3^n} + \frac{(-3)^n}{3^n} = 1 + (-1)^n$

- If n is even, $(-1)^n = 1$, so $U_n = 1+1 = 2$.
- If n is odd, $(-1)^n = -1$, so $U_n = 1 + (-1) = 0$.

Thus,

$$U_n = \begin{cases} 2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

$\implies (U_n)$ is divergent because it has two limits.

c. We have :

$$U_n = \left(1+\frac{2}{n}\right)^{\cdot} = \left(\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\right)^2$$

$$= \left(e^{\frac{n}{2}\ln\left(1+\frac{2}{n}\right)}\right)^2 . = \left(e^{\frac{\ln\left(1+\frac{2}{n}\right)}{\frac{2}{n}}}\right)^2 = \left(e^1\right)^2 = e^2.$$

with : $\lim_{n\to+\infty} \frac{\ln\left(1+\frac{2}{n}\right)}{\frac{2}{n}} = 1 = \lim_{X\to 0} \frac{\ln(1+X)}{X}$.

So, $\lim_{n\to+\infty} U_n = e^2 \implies (U_n)$ is convergent and converges to e^2

2. Let $n \in \mathbb{N}$. We have :

$$-1 \le \cos n \le 1 \text{ and } -1 \le \sin\sqrt{n} \le 1. \iff 1 \le 2+\cos n \le 3 \text{ and } 2 \le 3-\sin\sqrt{n} \le 4.$$

$$\iff \frac{1}{4} \le \frac{2+\cos n}{3-\sin\sqrt{n}} \le \frac{3}{2} \iff \frac{1}{4} \le U_n \le 2,$$

thus the sequence (u_n) is bounded.

Exercise 2 : Let (U_n) be defined by :

$$\begin{cases} U_0 = \frac{3}{2} \\ U_{n+1} = (U_n - 1)^2 + 1 \end{cases}$$

1. Show by induction resoning that $P(n) : \forall n \in \mathbb{N}, 1 < U_n < 2$.

- For $n = 0$, $1 < U_0 = \frac{3}{2} < 2 \implies P(0)$ is true
- Suppose $P(n) : 1 < U_n < 2$ is true and show that $P(n+1) : 1 < U_{n+1} < 2$ is true.

We have :
$1 < U_n < 2 \implies 0 < U_n - 1 < 1 \implies 1 < (U_n-1)^2 + 1 < 2 \implies 1 < U_{n+1} < 2$
Thus by induction reasoning : $1 < U_n < 2,\ \forall n \in \mathbb{N}$.

2. Study the monotonicity of (U_n).

$$U_{n+1} - U_n = (U_n - 1)^2 + 1 - U_n = U_n^2 - 2U_n + 1 + 1 - U_n$$
$$= U_n^2 - 3U_n + 2 = (U_n - 1)(U_n - 2)$$

But $\forall n$, $U_n - 1 > 0$ and $U_n - 2 < 0$ (from question 1), so,

$$U_{n+1} - U_n < 0$$

Hence, the sequence (U_n) is strictly decreasing.

3. Deduce that (U_n) is convergent.

We have :

$$\begin{cases} 1 < U_n < 2 \quad \text{(bounded below by 1)} \\ (U_n) \text{ is strictly decreasing} \end{cases} \implies U_n \text{is convergent.}$$

Since (U_n) is convergent,

$$\lim_{n \to +\infty} U_{n+1} = \lim_{n \to +\infty} U_n = \ell$$

then,

$$U_{n+1} = (U_n - 1)^2 + 1 \implies \ell = (\ell - 1)^2 + 1 \implies \ell^2 - 3\ell + 2 = 0$$

The solutions are $\ell = 1$ or $\ell = 2$. But $U_0 = \dfrac{3}{2}$ and the sequence is decreasing $\implies \ell = 1$.

Exercise 3 :$(U_n)_{n \in \mathbb{N}^*}$ defined by

$$U_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}.$$

Define the sequences : $V_n = U_{2n}$ and $W_n = U_{2n+1}$.

1. Show that (V_n) and (W_n) are adjacent.

• The monotonicity of (V_n) :

$$V_{n+1} - V_n = U_{2(n+1)} - U_{2n} = \sum_{k=1}^{2n+2} \frac{(-1)^{k+1}}{k} - \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k}$$
$$= \left(\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} + \frac{(-1)^{2n+2}}{2n+1} + \frac{(-1)^{2n+3}}{2n+2} \right) - \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k}$$
$$= \frac{(-1)^{2n+2}}{2n+1} + \frac{(-1)^{2n+3}}{2n+2} = \frac{1}{2n+1} + \frac{-1}{2n+2} = \frac{1}{(2n+1)(2n+2)} > 0$$

$\implies V_{n+1} - V_n > 0 \implies (V_n)$ is increasing.

• Monotonicity of (W_n) :

$$W_{n+1} - W_n = U_{2(n+1)+1} - U_{2n+1} = \sum_{k=1}^{2n+3} \frac{(-1)^{k+1}}{k} - \sum_{k=1}^{2n+1} \frac{(-1)^{k+1}}{k}$$
$$= \left(\sum_{k=1}^{2n+1} \frac{(-1)^{k+1}}{k} + \frac{(-1)^{2n+3}}{2n+2} + \frac{(-1)^{2n+4}}{2n+3} \right) - \sum_{k=1}^{2n+1} \frac{(-1)^{k+1}}{k}$$
$$= \frac{(-1)^{2n+3}}{2n+2} + \frac{(-1)^{2n+4}}{2n+3} = \frac{-1}{2n+2} + \frac{1}{2n+3} = \frac{-1}{(2n+3)(2n+2)} < 0$$

$\implies W_{n+1} - W_n > 0 \implies (W_n)$ is strictly decreasing.

• Calculate $\lim\limits_{n\to+\infty}(W_n - V_n)$:

$$\begin{aligned}W_n - V_n &= \sum_{k=1}^{2n+1}\frac{(-1)^{k+1}}{k} - \sum_{k=1}^{2n}\frac{(-1)^{k+1}}{k}\\ &= \left(\sum_{k=1}^{2n}\frac{(-1)^{k+1}}{k} + \frac{(-1)^{2n+2}}{2n+1}\right) - \sum_{k=1}^{2n}\frac{(-1)^{k+1}}{k}\\ &= \frac{(-1)^{2n+2}}{2n+1} = \frac{1}{2n+1}\end{aligned}$$

Therefore, $\lim\limits_{n\to+\infty}(W_n - V_n) = 0 \implies (V_n)$ and (W_n) are adjacent sequences.

• Deduce that (U_n) is convergent :
The sequences (V_n) and (W_n) are two subsequences of (U_n) and both converge to the same limit. Thus, (U_n) is convergent with the same limit.

<u>Exercise 4</u> : We have $\forall n \in \mathbb{N}$.

$$\begin{cases} U_0 = 2, \quad U_1 = \dfrac{4}{9}, \\ U_{n+2} = \dfrac{1}{27}(12U_{n+1} - U_n), \end{cases}$$

and $V_n = U_n - \dfrac{1}{3^n}$.

1. Let us show that : $P(n) : \forall n \in \mathbb{N} : U_{n+1} = \dfrac{1}{9}U_n + \dfrac{2}{3^{n+2}}$.
We use reasoning by induction for $n \in \mathbb{N}$:

— For $n = 0$, we have :

$$U_1 = \frac{1}{9}U_0 + \frac{2}{3^2} \implies \frac{4}{9} = \frac{1}{9}\cdot 2 + \frac{2}{9} \implies \frac{4}{9} = \frac{4}{9},$$

so $P(0)$ is true.

— We suppose that $P(n)$ is true for $n \geq 0$, that is :

$$U_{n+1} = \frac{1}{9}U_n + \frac{2}{3^{n+2}} \implies U_n = 9U_{n+1} - \frac{2}{3^n}.$$

We now demonstrate that $P(n+1) : U_{n+2} = \dfrac{1}{9}U_{n+1} + \dfrac{2}{3^{n+3}}$ is true.
We have :

$$\begin{aligned}U_{n+2} &= \frac{1}{27}\left(12U_{n+1} - U_n\right) = \frac{4}{9}U_{n+1} - \frac{1}{27}U_n\\ &= \frac{4}{9}U_{n+1} - \frac{1}{27}U_n = \frac{4}{9}U_{n+1} - \frac{1}{27}\left(\underbrace{9U_{n+1} - \frac{2}{3^n}}_{\text{by the hypothesis}}\right)\\ &= \frac{4}{9}U_{n+1} - \frac{1}{3}U_{n+1} + \frac{2}{3^{n+3}} = \frac{1}{9}U_{n+1} + \frac{2}{3^{n+3}}.\end{aligned}$$

Thus, $P(n+1)$ is true $\implies P(n)$ is true for all $n \in \mathbb{N}$. Then,

$$\forall n \in \mathbb{N} : U_{n+1} = \frac{1}{9}U_n + \frac{2}{3^{n+2}}$$

2. Let us show that the sequence (V_n) is a geometric sequence.

$$V_{n+1} = U_{n+1} - \frac{1}{3^{n+1}} = \left(\frac{1}{9}U_n + \frac{2}{3^{n+2}}\right) - \frac{1}{3^{n+1}} = \frac{1}{9}U_n - \frac{1}{3^{n+2}} = \frac{1}{9}\left(U_n - \frac{1}{3^n}\right) = \frac{1}{9}V_n.$$

Thus, we have : $V_{n+1} = \frac{1}{9}V_n$, so (V_n) is a geometric sequence with ratio $q = \frac{1}{9}$ and the first term :

$$V_0 = U_0 - \frac{1}{3^0} = 2 - 1 = 1.$$

•Let us express U_n in terms of n.
(V_n) is a geometric sequence, so :

$$V_n = V_0 q^n \quad \Rightarrow \quad V_n = \left(\frac{1}{9}\right)^n \Longrightarrow V_n = \frac{1}{3^{2n}}, \quad n \in \mathbb{N}.$$

Now, we have :

$$V_n = U_n - \frac{1}{3^n} \quad \Rightarrow \quad U_n = V_n + \frac{1}{3^n} = \frac{1}{3^{2n}} + \frac{1}{3^n} = \frac{3^n + 1}{3^{2n}} \quad n \in \mathbb{N}.$$

3. Express in terms of n, the sum

$$S_n = \sum_{k=0}^{n} U_k.$$

We have :

$$S_n = \sum_{k=0}^{n} U_k = \sum_{k=0}^{n}\left(V_k + \frac{1}{3^k}\right) = \sum_{k=0}^{n} V_k + \sum_{k=0}^{n}\frac{1}{3^k}.$$

This simplifies to :

$$S_n = V_0\frac{1-\left(\frac{1}{9}\right)^{n+1}}{1-\frac{1}{9}} + \frac{1-\left(\frac{1}{3}\right)^{n+1}}{1-\frac{1}{3}}.$$

Simplifying further :

$$S_n = \frac{1-\left(\frac{1}{9}\right)^{n+1}}{\frac{8}{9}} + \frac{1-\left(\frac{1}{3}\right)^{n+1}}{\frac{2}{3}}.$$

Thus :

$$S_n = \frac{9}{8} - \frac{9}{8}\left(\frac{1}{9}\right)^{n+1} + \frac{3}{2} - \frac{3}{2}\left(\frac{1}{3}\right)^{n+1}.$$

Finally, we obtain :

$$S_n = \frac{21}{8} - \frac{1}{8\cdot 3^{2n}} - \frac{1}{2\cdot 3^n} = \frac{7\cdot 3^{2n+1} - 4\cdot 3^n - 1}{8\cdot 3^{2n}}.$$

Remark.

$\sum_{k=0}^{n} V_k$: is a sum of $(n+1)$ terms of a geometric sequence with ratio $q = \frac{1}{9}$ and first term 1.

$\sum_{k=0}^{n} \frac{1}{3k}$: is a sum of $(n+1)$ terms of a geometric sequence with ratio $q = \frac{1}{3}$ and first term 1.