Contents

3 Real functions of real variables **1**

3.4 Logarithmic, exponential and power functions 1

3.4.1 Logarithmic function 1

3.4.2 Exponential function 2

3.4.3 Power function 3

3.5 Trigonometric functions 3

3.5.1 Sine and cosine functions 3

3.5.2 Tangent and cotangent functions 4

3.5.3 Inverse trigonometric functions 5

3.6 Hyperbolic and inverse hyperbolic functions 8

3.6.1 Hyperbolic cosine and its inverse 8

3.6.2 Hyperbolic sine and its inverse 8

Chapter 3

Real functions of real variables

3.4 Logarithmic, exponential and power functions

3.4.1 Logarithmic function

Definition 3.1. We call the natural logarithm and we denote by ln the function that equals 0 at 1 and the unique primitive of the function $x \to \frac{1}{x}$ defined on $\mathbb{R}_+^*$,

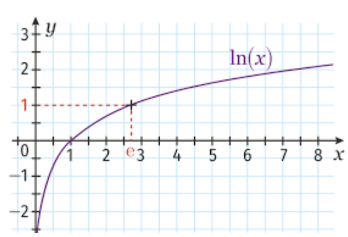
$$\begin{aligned} ln :]0,+\infty[&\to \mathbb{R} \\ x &\to \ln(x) \end{aligned}$$

Properties 3.1. (Properties of logarithms)

Let a and b be strictly positive real numbers, and n is a real number:

1. $\ln(a \times b) = \ln a + \ln b$,
2. $\ln(\frac{a}{b}) = \ln a - \ln b$,
3. $\ln(\frac{1}{a}) = -\ln a$,
4. $\ln(a^n) = n \ln a$, $(for\ all\ n \in \mathbb{N})$.

The function $x \to \ln(x)$ is a continuous, strictly increasing function and defines a bijection of $]0,+\infty[$ on $\mathbb{R}$.



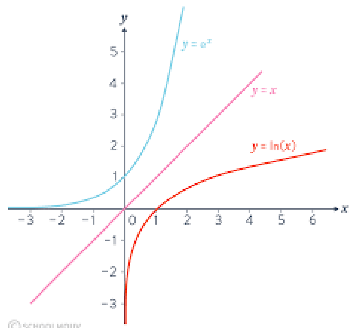
Particular limits:

$$\lim_{x\to+\infty} \ln(x) = +\infty, \qquad \lim_{x\to 0^+} \ln(x) = -\infty, \qquad \lim_{x\to 0^+} x\ln(x) = 0,$$
$$\lim_{x\to+\infty} \frac{\ln(x)}{x} = 0, \qquad \lim_{x\to 0^+} \frac{\ln(x+1)}{x} = 1.$$

3.4.2 Exponential function

Definition 3.2. The inverse function of $\ln :]0, +\infty[\to \mathbb{R}$ is called the exponential function, noted $\exp : \mathbb{R} \to]0, +\infty[$.

For $x \in \mathbb{R}$, we also note e^x for $\exp(x)$.



Proposition 3.1. The exponential function satisfies the following properties:

1. $e^{(\ln x)} = x, \forall x \in]0, +\infty[$, and $\ln(e^x) = x, \forall x \in \mathbb{R}$,
2. $e^{(a+b)} = e^a \times e^b$,
3. $(e^a)^n = e^{na}$,
4. $e^{-a} = \frac{1}{e^a}$,
5. $\exp : \mathbb{R} \to]0, +\infty[$ is continuous and strictly increasing function.
6. The exponential function is differentiable and $(e^x)' = e^x$, $\forall x \in \mathbb{R}$.

Particular limits:

$$\lim_{x\to+\infty} e^x = +\infty, \qquad \lim_{x\to-\infty} e^x = 0, \qquad \lim_{x\to+\infty} xe^x = +\infty,$$
$$\lim_{x\to+\infty} \frac{e^x}{x} = +\infty, \qquad \lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

3.4.3 Power function

Definition 3.3. The **power function** of a positive real a is the function f_a defined on $\mathbb{R}$ by:

$$f_a(x) = a^x = e^{x \ln a}.$$

Remark 3.1. The power function is strictly positive,

$$\forall x \in \mathbb{R}, a^x = e^{x \ln a} > 0.$$

Properties 3.2. For all $a, b > 0$, we have the following equalities:

1. $\forall x, y \in \mathbb{R}$, $a^{x+y} = a^x \times a^y$ and $a^{x-y} = \frac{a^x}{a^y}$,
2. $\forall x, y \in \mathbb{R}$, $(a^x)^y = a^{xy}$,
3. $\forall x \in \mathbb{R}$, $(ab)^x = a^x \times b^x$.

Study of the power function:
Let be the function f_a defined on $\mathbb{R}$ by: $f_a(x) = a^x$:
As $a^x = e^{x \ln a}$, it is continuous and differentiable on $\mathbb{R}$; because it is a composition of continuous and differentiable functions on $\mathbb{R}$. So:

$$\forall x \in \mathbb{R}, \; f_a'(x) = (e^{x \ln a})' = \ln(a) e^{x \ln a} = \ln(a) a^x.$$

The sign of the derivative therefore depends on the sign of $\ln(a)$. Then:

1. If $a > 1$, then $\forall x \in \mathbb{R}, \; f_a'(x) > 0$, the power function is increasing.
2. If $0 < a < 1$, then $\forall x \in \mathbb{R}, \; f_a'(x) < 0$, the power function is decreasing.

Particular limits:

$a > 1$	$0 < a < 1$
$\lim_{x \to +\infty} a^x = +\infty$	$\lim_{x \to +\infty} a^x = 0$
$\lim_{x \to -\infty} a^x = 0$	$\lim_{x \to -\infty} a^x = +\infty$

3.5 Trigonometric functions

3.5.1 Sine and cosine functions

Function	$\sin x$	$\cos x$
Definition domain	$\mathbb{R}$	$\mathbb{R}$
Parity	Odd	even
Period	$T = 2\pi$	$T = 2\pi$
Derivative	$\cos x$	$-\sin x$

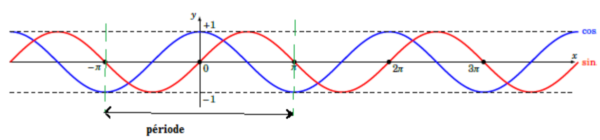
Properties 3.3. The **sine** and **cosine** functions satisfy the following properties, for all $x \in \mathbb{R}$:

- $\cos^2 x + \sin^2 x = 1$,
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$,
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$,
- $\sin 2x = 2 \sin x \cos x$,
- $\cos 2x = \cos^2 x - \sin^2 x = -1 + 2\cos^2 x = 1 - 2\sin^2 x$.

Addition formula:
$\forall a, b \in \mathbb{R}$, we have :

$$\begin{aligned} \sin(a+b) &= \sin a \cos b + \sin b \cos a \\ \sin(a-b) &= \sin a \cos b - \sin b \cos a \\ \cos(a+b) &= \cos a \cos b - \sin a \sin b \\ \cos(a-b) &= \cos a \cos b + \sin a \sin b \end{aligned}$$

Variations:
The sine and cosine functions are continuous and differentiable on $\mathbb{R}$. Since they are periodic, with period 2π, we can restrict the domain of the study to the interval of length 2π, for example $[-\pi, \pi]$.



3.5.2 Tangent and cotangent functions

Definition 3.4. We call **tangent** the function tan (or tg) defined by:

$$x \longmapsto \tan x = \frac{\sin x}{\cos x}, \ \textit{for all } x \in \mathbb{R} \backslash \{A\}$$

where $A = \left\{ \frac{\pi}{2} + k\pi / \ k \in \mathbb{Z} \right\}$.

We call **cotangent** the function cot defined by:

$$x \longmapsto \cot x = \frac{\cos x}{\sin x}, \ \textit{for all } x \in \mathbb{R} \backslash \{B\}$$

where $B = \{k\pi /\ k \in \mathbb{Z}\}$.

For all $x \in \mathbb{R}\backslash \{A \cup B\}$, we have the equality

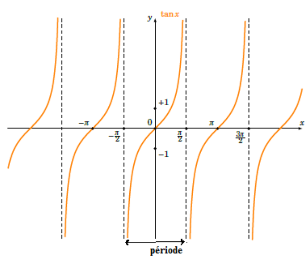
$$\cot x = \frac{1}{\tan x}.$$

Derivatives and variations:

The tangent and cotangent functions are continuous and differentiable over their defined domains and we have:

$$\begin{aligned}(\tan x)' &= \frac{1}{\cos^2 x} = 1 + \tan^2 x,\ \forall\ x \in \mathbb{R}\backslash\{A\} \\ (\cot x)' &= -\frac{1}{\sin^2 x} = -(1 + \cot^2 x),\ \forall x \in \mathbb{R}\backslash\{B\}.\end{aligned}$$

The two functions being periodic with period π , we can therefore restrict the domain of the study to an interval of length π, for example $\left]\frac{-\pi}{2}, \frac{\pi}{2}\right[$for the tangent and $]0, \pi[$ for the cotangent.



3.5.3 Inverse trigonometric functions

Arccosine function

Consider the cosine function

$$\begin{aligned}\cos \quad &: \quad \mathbb{R} \to [-1, 1] \\ x \quad &\mapsto \quad \cos x.\end{aligned}$$

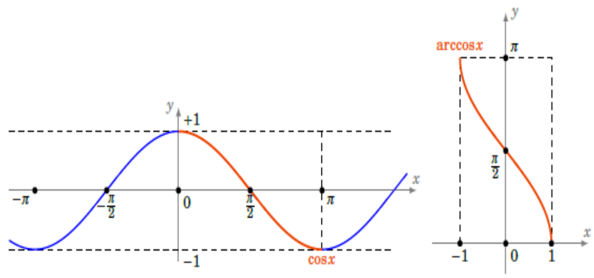
To obtain a bijection from this function, we must consider the restriction of cosine at interval $[0, \pi]$. On this interval the cosine function is continuous and strictly decreasing, therefore

$$\cos : [0, \pi] \to [-1, 1]$$

is a bijection. Its inverse function (bijection) is the **arccosine** function:

$$\arccos [-1, 1] :\to [0, \pi].$$

Thus, by definition of inverse function, we have



1. $\cos(\arccos x) = x, \forall x \in [-1, 1]$,
2. $\arccos(\cos x) = x, \forall x \in [0, \pi]$.
3. If $x \in [0, \pi]$, $\cos x = y \Leftrightarrow x = \arccos y$.
4. $\sin(\arccos x) = \sqrt{1 - x^2}, \forall x \in [-1, 1]$.

The derivative of arccos:

$$(\arccos x)' = \frac{-1}{\sqrt{1 - x^2}}, \ \forall x \in \left]-1, 1\right[.$$

Arcsine function

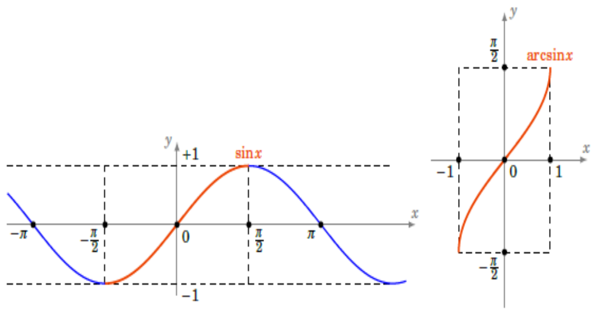
The sine function has a strictly positive derivative function on $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, so it is a bijection of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ on $[-1, 1]$. The inverse of sine function is called an **arcsine** function and is denoted arcsin,

$$\begin{aligned} \arcsin \quad &: \quad [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ x \quad &\mapsto \quad \arcsin x. \end{aligned}$$

Properties 3.4.

1. $\sin(\arcsin x) = x, \forall x \in [-1, 1]$,
2. $\arcsin(\sin x) = x, \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,
3. $\sin x = y \Leftrightarrow x = \arcsin y, \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,
4. $\cos(\arcsin x) = \sqrt{1 - x^2}, \forall x \in [-1, 1]$,
5. The arcsin function is derivable on $]-1, 1[$, and we have

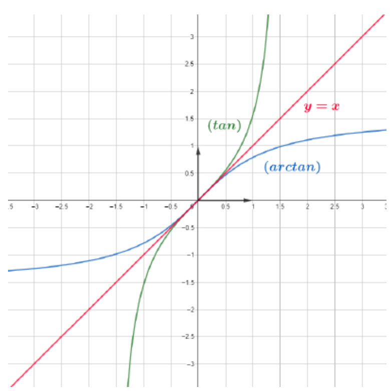
$$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}.$$



Arctangent function

The tangent function has a strictly positive derivative function on $]-\frac{\pi}{2}, \frac{\pi}{2}[$, so it is a bijection of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ on $\mathbb{R}$. The reciprocal bijection is called the **arctangent** function and is denoted by arctan

$$\begin{aligned} \arctan \quad : \quad & \mathbb{R} \to]-\frac{\pi}{2}, +\frac{\pi}{2}[\\ & x \to \arctan(x) \end{aligned}$$



Properties 3.5. 1. $\tan(\arctan(x)) = x, \forall x \in \mathbb{R}$,

2. $\arctan(\tan(x)) = x, \forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$,

3. $\tan(x) \ = y \Leftrightarrow x = \arctan\, y,\ \forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$,

4. The arctan function is derivable on $\mathbb{R}$, and we have

$$(\arctan(x))' = \frac{1}{1+x^2}$$

3.6 Hyperbolic and inverse hyperbolic functions

3.6.1 Hyperbolic cosine and its inverse

Definition 3.5. For $x \in \mathbb{R}$, **the hyperbolic cosine** is given by:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

The function $\cosh : [0, +\infty[\to [1, +\infty[$ is a bijection.
Its inverse function is $Argch : [1, +\infty[\to [0, +\infty[$. (Hyperbolic cosine argument)

Properties 3.6.

- $\forall\ x \in [1, +\infty[$, $\cosh(Argch(x)) = x$.
- $\forall\ x \in [0, +\infty[$, $Argch(\cosh(x)) = x$.
- $\forall\ x \in [1, +\infty[$, $Argch(x) = \ln(\sqrt{x^2 - 1} + x)$.
- The function $Argch$ is continuous on $[1, +\infty[$, differentiable on $]1, +\infty[$ and

$$(Argch(x))' = \frac{1}{\sqrt{x^2 - 1}}.$$

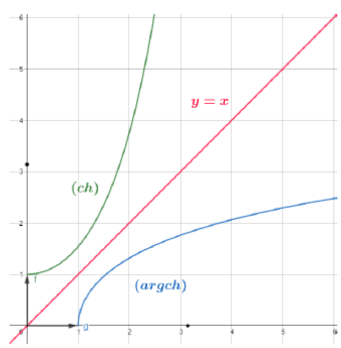


Figure 3.1: Hyperbolic cosine and hyperbolic cosine argument

3.6.2 Hyperbolic sine and its inverse

Definition 3.6. For $x \in \mathbb{R}$, **the hyperbolic sine** is given by:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$\sinh : \mathbb{R} \to \mathbb{R}$ is a continuous, differentiable, strictly increasing function verifying

$$\lim_{x \to +\infty} \sinh x = +\infty \text{ and } \lim_{x \to -\infty} \sinh x = -\infty,$$

therefore, sinh is a bijection.
The inverse function is $Argsh : \mathbb{R} \to \mathbb{R}$. (Hyperbolic sine argument)

Properties 3.7.

- $\forall\, x \in \mathbb{R}$, $\sinh(Argsh(x)) = x$ and $Argsh(\sinh(x)) = x$.
- $\cosh^2 x - \sinh^2 x = 1$.
- $(\cosh x)' = \sinh x$, $(\sinh x)' = \cosh x$.
- $Argsh : \mathbb{R} \to \mathbb{R}$ is strictly increasing and continuous function.
- $Argsh(x) = \ln(x + \sqrt{x^2+1})$.
- $Argsh$ is differentiable and $(Argsh(x))' = \frac{1}{\sqrt{x^2+1}}$.

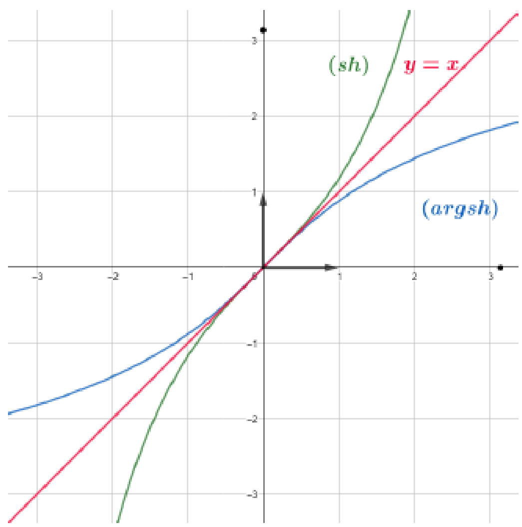


Figure 3.2: Hyperbolic sine and hyperbolic sine argument