Solution of Series 3

Exercise 1 :

1. $\lim\limits_{x\to 0}\dfrac{\ln(1+x^2)}{\sin^2 x}=\lim\limits_{x\to 0}\dfrac{\frac{\ln(1+x^2)}{x^2}}{\frac{\sin^2 x}{x^2}}=\dfrac{1}{1}=1$
2. $\lim\limits_{x\to 0}\dfrac{x\sin x}{1-\cos x}=\lim\limits_{x\to 0}\dfrac{(x\sin x)(1+\cos x)}{(1-\cos x)(1+\cos x)}=\lim\limits_{x\to 0}\dfrac{(x\sin x)(1+\cos x)}{(1-\cos^2 x)}=\lim\limits_{x\to 0}\dfrac{(x\sin x)(1+\cos x)}{\sin^2 x}$
$$=\lim_{x\to 0}\frac{(x)(1+\cos x)}{\sin x}=\lim_{x\to 0}\frac{x}{\sin x}(1+\cos x)$$
$$=\lim_{x\to 0}\frac{1}{\frac{\sin x}{x}}(1+\cos x)=1\times 2=2$$
3. $\lim\limits_{x\to 0}x\exp\left(\dfrac{1}{x}-1\right)=\lim\limits_{x\to 0}x\exp\left(\dfrac{1}{x}\right)\exp(-1)=\lim\limits_{t\to +\infty}e^{-1}\dfrac{\exp t}{t}=+\infty\quad\left(t=\dfrac{1}{x},x\to 0,t\to +\infty\right)$
4.
$$\lim_{x\to +\infty}\left(\frac{x}{x-2}\right)^x=\lim_{x\to +\infty}\left(1+\frac{1}{\frac{x-2}{2}}\right)^x=\lim_{x\to +\infty}\left(\left(1+\frac{1}{\frac{x-2}{2}}\right)^x\right)^{\frac{x-2}{2}\times\frac{2}{x-2}}$$
$$=\lim_{x\to +\infty}\left(\left(1+\frac{1}{\frac{x-2}{2}}\right)^{\frac{x-2}{2}}\right)^{\frac{2x}{x-2}}=\lim_{x\to +\infty}e^{\frac{2x}{x-2}}=e^2$$

Exercise 2 :

$$f(x)=\begin{cases}\cos^2(\pi x), & x\le 1\\ 1+\dfrac{\ln x}{x}, & x>1\end{cases}$$

a. The continuity of f on $\mathbb{R}$:

On $]-\infty,1]$ f is continuous (because f is the product of two continuous functions).
On $]1,+\infty[$ f is continuous (because f is the sum of two continuous functions).
Study the continuity of f at $x=1$:
we have

$$\lim_{x\to 1^+}f(x)=\lim_{x\to 1^+}\left(1+\frac{\ln x}{x}\right)=1=f(1)$$
$$\lim_{x\to 1^-}f(x)=\lim_{x\to 1^-}\left(\cos^2(\pi x)\right)=1=f(1)$$

So f is continuous at $x=1$ because $\lim\limits_{x\to 1^+}f(x)=\lim\limits_{x\to 1^-}f(x)=f(1)\Longrightarrow f$ is continuous on $\mathbb{R}$.

b. The differentiability of f on $\mathbb{R}$:

On $]-\infty,1]$ f is differentiable (because f is the product of two differentiable functions).
On $]1,+\infty[$ f is differentiable (because f is the sum of two differentiable functions).
Study the differentiability of f at $x=1$:
we have

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^-}\frac{\cos^2(\pi x)-1}{x-1}=\frac{0}{0}\ (I.F)$$
$$=\lim_{x\to 1^-}\frac{\left(\cos^2(\pi x)-1\right)'}{(x-1)'}=\lim_{x\to 1^-}\frac{-2\pi\sin(\pi x)\cos(\pi x)}{1}=0$$
$$\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{1+\frac{\ln x}{x}-1}{x-1}=\frac{0}{0}\ (I.F)$$
$$=\lim_{x\to 1^+}\frac{(\ln x)'}{((x-1)x)'}=\lim_{x\to 1^+}\frac{1}{(2x-1)x}=1$$

Then $\lim_{x\to 1^-}\dfrac{f(x)-f(1)}{x-1} \neq \lim_{x\to 1^+}\dfrac{f(x)-f(1)}{x-1} \Longrightarrow f$ is not differentiable at $x=1 \Longrightarrow f$ is not differentiable on $\mathbb{R}$.

2. We conclude that : f is continuous on $\mathbb{R}$ but it is not differentiable on $\mathbb{R} \Longrightarrow f$ is not of class C^1.

Exercise 3 :

$$\begin{aligned} f(x) &= x^2\cos\frac{1}{x} \\ D_f &=]-\infty,0[\cup]0,+\infty[\end{aligned}$$

1. f is extendable by continuity at $x=0$? Calculate $\lim_{x\to 0} f(x)$

we have

$$\begin{aligned} -1 &\le \cos\frac{1}{x} \le 1 \Longrightarrow \lim_{x\to 0}\left(-x^2\right) \le \lim_{x\to 0}\left(x^2\cos\frac{1}{x}\right) \le \lim_{x\to 0}\left(x^2\right) \\ &\Longrightarrow \lim_{x\to 0} x^2\cos\frac{1}{x} = 0 \qquad \text{(by squeeze theorem)} \end{aligned}$$

then f is extendable by continuity at $x=0$ and its continuity extension function is :

$$\widetilde{f}(x) = \begin{cases} x^2\cos\dfrac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$$

2. Let' show that $f(x)-1=0$ admits a solution on $\left]\dfrac{3}{\pi},\dfrac{4}{\pi}\right[$,

we set

$$g(x) = f(x)-1$$

the function g is continuous on $\left]\dfrac{3}{\pi},\dfrac{4}{\pi}\right[$ and, also

$$\begin{aligned} g\left(\frac{3}{\pi}\right) &= f\left(\frac{3}{\pi}\right) - 1 = \frac{9}{2\pi^2} - 1 < 0 \\ g\left(\frac{4}{\pi}\right) &= f\left(\frac{3}{\pi}\right) - 1 = \frac{8\sqrt{2}}{\pi^2} - 1 > 0 \end{aligned}$$

according to the Intermediate value Theorem, there exists at least one solution $c \in \left]\dfrac{3}{\pi},\dfrac{4}{\pi}\right[$ such that $g(c)=0$

Is this solution unique ?

we have

$$\begin{aligned} g'(x) &= (f(x)-1)' = \left(x^2\cos\frac{1}{x}\right)' = 2x\cos\frac{1}{x} + x^2\left(-\frac{1}{x^2}\right)\left(-\sin\frac{1}{x}\right) \\ &= 2x\cos\frac{1}{x} + \sin\frac{1}{x} > 0 \end{aligned}$$

Then g is strictly increasing on $\left]\dfrac{3}{\pi},\dfrac{4}{\pi}\right[$ and $g\left(\dfrac{3}{\pi}\right)\times g\left(\dfrac{4}{\pi}\right) < 0$ then $\exists! c \in \left]\dfrac{3}{\pi},\dfrac{4}{\pi}\right[$ such that $g(c)=0$ (by Intermediate Value Theorem), the equation $f(x)-1=0$ admits a unique real solution on $\left]\dfrac{3}{\pi},\dfrac{4}{\pi}\right[$.

Exercise 4 :

1.

$$\begin{aligned}
f(x) &= \arccos(2x-1) - \arcsin(3x^2) \\
D_f &= \left\{x \in \mathbb{R} : -1 \leq 2x-1 \leq 1 \text{ and } -1 \leq 3x^2 \leq 1\right\} \\
&\Leftrightarrow -1 \leq 2x-1 \leq 1 \quad \text{and} \quad -1 \leq 3x^2 \leq 1 \\
&\Leftrightarrow 0 \leq x \leq 1 \qquad \text{and} \qquad x^2 \leq \frac{1}{3} \\
&\Leftrightarrow x \in \ [0,1] \qquad \text{and} \qquad x \in \left[\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right] \\
&\Leftrightarrow x \in \left[0, \frac{1}{\sqrt{3}}\right] \\
&\Leftrightarrow D_f = \left[0, \frac{1}{\sqrt{3}}\right]
\end{aligned}$$

2. We calculate f', we have $(\arccos U)' = \dfrac{-U'}{\sqrt{1-U^2}}, (\arcsin U)' = \dfrac{U'}{\sqrt{1-U^2}}$

$$\begin{aligned}
f'(x) &= \frac{-(2x-1)'}{\sqrt{1-(2x-1)^2}} - \frac{(3x^2)'}{\sqrt{1-(3x^2)^2}} \\
&= \frac{-2}{\sqrt{1-(2x-1)^2}} - \frac{6x}{\sqrt{1-(3x^2)^2}}
\end{aligned}$$

Exercise 5 :

1. Solve the equation

$$\arcsin x = \arcsin\frac{2}{5} + \arcsin\frac{3}{5}$$

with

$$\begin{aligned}
\cos(\arcsin x) &= \sqrt{1-x^2} \quad \text{and} \\
\sin(a+b) &= \sin a \cos b + \sin b \cos a
\end{aligned}$$

we have

$$\begin{aligned}
\sin(\arcsin x) &= \sin\left(\arcsin\frac{2}{5} + \arcsin\frac{3}{5}\right) \\
x &= \sin\left(\arcsin\frac{2}{5}\right)\cos\left(\arcsin\frac{3}{5}\right) + \sin\left(\arcsin\frac{3}{5}\right)\cos\left(\arcsin\frac{2}{5}\right) \\
&= \frac{2}{5}\cos\left(\arcsin\frac{3}{5}\right) + \frac{3}{5}\cos\left(\arcsin\frac{2}{5}\right) \\
x &= \frac{2}{5}\sqrt{1-\left(\frac{3}{5}\right)^2} + \frac{3}{5}\sqrt{1-\left(\frac{2}{5}\right)^2} = \frac{3\sqrt{21}+8}{25}
\end{aligned}$$

2. Let

$$g(x) = \arctan x + \arctan\frac{1}{x}, \qquad for\ all \quad x \in\,]0,+\infty[$$

$$g'(x) = \frac{1}{1+x^2} + \frac{-\frac{1}{x^2}}{1+\left(\frac{1}{x^2}\right)^2} = 0$$

So g is constant on $]0,+\infty[$, hence

$$g(x)=g(1)=2\arctan(1)=2\left(\frac{\pi}{4}\right)=\frac{\pi}{2}$$

$$\Rightarrow \arctan x+\arctan\frac{1}{x}=\frac{\pi}{2}$$

Exercise 6 :

1.

$$\begin{aligned} sh\left(\arg chx\right) &= ? \\ we\ hawe\ ch^2\alpha-sh^2\alpha &= 1\ \ with\ \ \alpha=\arg chx \end{aligned}$$

$$\begin{aligned} sh^2\left(\arg chx\right) &= ch^2\left(\arg chx\right)-1 \\ &= x^2-1 \\ &\Longrightarrow sh\left(\arg chx\right)=\pm\sqrt{x^2-1} \end{aligned}$$

with $\arg chx\geq 0$ then $sh\left(\arg chx\right)\geq 0$ then $sh\left(\arg chx\right)=\sqrt{x^2-1}$

2.

$$\frac{2ch^2x-sh\left(2x\right)}{x-\ln\left(chx\right)-\ln 2}$$

we have

$$\begin{aligned} 2ch^2x-sh2x &= 2\left(\frac{e^x+e^{-x}}{2}\right)^2-\left(\frac{e^{2x}-e^{-2x}}{2}\right) \\ &= e^{-2x}+1 \end{aligned}$$

and

$$\begin{aligned} x-\ln\left(chx\right)-\ln 2 &= x-\ln\left(\frac{e^x+e^{-x}}{2}\right)-\ln 2 \\ &= x-\ln\left(e^x+e^{-x}\right)+\ln 2-\ln 2 \\ &= x-\ln\left(e^x\left(e^{-2x}+1\right)\right) \\ &= x-\ln\left(e^x\right)-\ln\left(e^{-2x}+1\right) \\ &= -\ln\left(e^{-2x}+1\right) \end{aligned}$$

then

$$\frac{2ch^2x-sh\left(2x\right)}{x-\ln\left(chx\right)-\ln 2}=-\frac{e^{-2x}+1}{\ln\left(e^{-2x}+1\right)}$$

3.

$$\cos\left(\arctan x\right)=?$$

we have

$$\frac{1}{\cos^2\alpha}=1+\tan^2\alpha\quad\text{then}\ \ \cos^2\alpha=\frac{1}{1+\tan^2\alpha}$$

for $\alpha=\left(\arctan x\right), x\in\mathbb{R}$

$$\begin{aligned} \cos^2\left(\arctan x\right) &= \frac{1}{1+\tan^2\left(\arctan x\right)}=\frac{1}{1+x^2} \\ &\Longrightarrow \cos\left(\arctan x\right)=\pm\sqrt{\frac{1}{1+x^2}} \end{aligned}$$

with $\arctan x\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, $\cos\left(\arctan x\right)\geq 0$ then

$$\cos\left(\arctan x\right)=\sqrt{\frac{1}{1+x^2}}$$