2.2 - Relationship between electrostatic field and potential

The electrostatic potential was defined from the elementary circulation of the field $\vec{E}$:

$$dC = \vec{E}.d\vec{r} = -dV$$

Or,

$$dV = \overrightarrow{grad}V.d\vec{r}$$

d'où la relation entre $\vec{E}$ et V :

$$\boxed{\vec{E}(M) = -\overrightarrow{grad}V(M) \quad \text{: relation locale} \qquad (6)}$$

The electrostatic field $\vec{E}$ derivative of the scalar potential V. Through this local relationship, which links the electrostatic field $\vec{E}$ and the electrostatic potential V; knowing V at a point in space is sufficient for determining the field. This relationship implies conditions of continuity and differentiability on the function V(M). The unit of electrostatic potential in the MKSA system is the volt (V). According to the relationship that links the electrostatic field $\vec{E}$ and the electrostatic potential V, the unit of electrostatic field is the Volt per meter (V/m).

2.3 – Properties

The CAB traffic of the field $\vec{E}$ along contour AB is:

$$\boxed{C_{AB} = \int_A^B \vec{E}.d\vec{r} = -\int_A^B dV = V(A) - V(B) = \frac{q}{4\Pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) \qquad (7)}$$

The circulation of the vector field $\vec{E}$, along AB, is therefore equal to the potential difference VA–VB. Thus, knowledge of $\vec{E}$ only defines potential differences. To obtain the potential at a point, an arbitrary origin of potentials must be defined. It is convenient to choose a zero potential at infinity when the charge distribution is limited to a finite domain. The circulation of the vector field $\vec{E}$, along AB is independent of the shape of the contour AB; it does not depend on the path followed. Consequently, the circulation of $\vec{E}$ is zero along any closed contour. The field $\vec{E}$ is a conservatively circulating vector field derived from a scalar function called the electrostatic potential. In summary:

$$C_{AB} = \int_{AB} \vec{E}.d\vec{r} = -\int_{A}^{B} dV = V(A) - V(B) \Leftrightarrow \oint \vec{E}.d\vec{r} = 0 \Leftrightarrow \vec{E} = -\overrightarrow{grad}V \qquad (8)$$

2.4 - Topography of an electric field

a) Field lines

To get an idea of what the field looks like $\vec{E}$ We draw the field lines, that is, the curves tangent at each point to the vector E defined at that point. These curves are conventionally oriented in the direction of the vector (Figure 3).

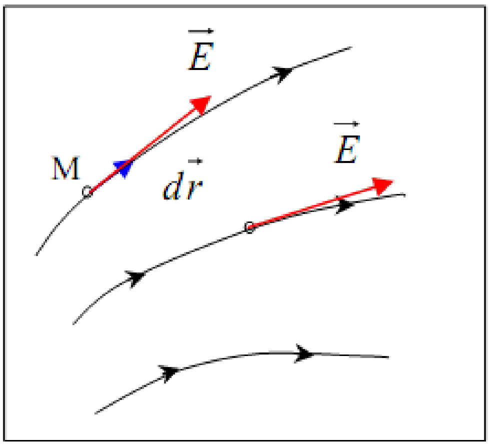

Figure 3

Let M be a point on a field line and $d\vec{r}$ the elementary displacement vector on a field line (Figure 3) Since $\vec{E}$ And $d\vec{r}$ are collinear, we have:

$$d\vec{r} \wedge \vec{E} = \vec{0} \qquad (9)$$

This relationship allows us to obtain the equations of the field lines. In the Cartesian coordinate system, let us define:

$$\vec{E} = E_x \vec{i} + E_y \vec{j} + E_z \vec{k} \text{ et } d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

Equation (9) leads to:

$$\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z} \qquad (10)$$

Example :

Consider a point charge at O. The lines of the field created by the point charge are half-lines concurring at O, diverging if q > 0 (figure 4-a) and converging if q < 0 (figure 4-b).

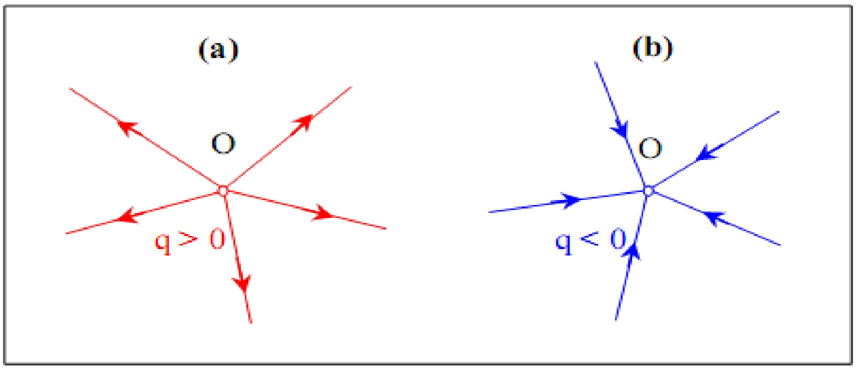

Figure 4

• Note that in a region where the field $\vec{E}$ is a well-defined and non-zero vector, we can follow a field line continuously.

• Two field lines cannot cross: Figure 4 shows that field lines start (Figure 4-a) or stop (Figure 4-b) at charges which are singular points.

b) Field tube

The set of field lines based on a closed contour constitutes a field tube (Figure 5).

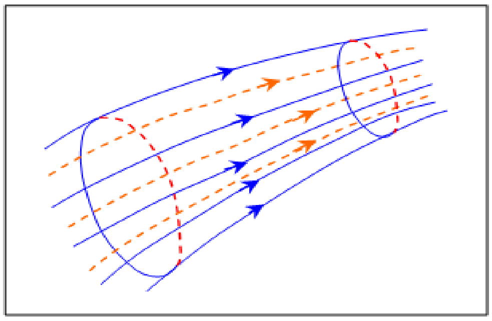

Figure 5

c) Equipotential surfaces

These are surfaces with equation V = constant, that is to say of equal potential (Figure 6).

According to the relationship $\vec{E} = -\overrightarrow{gradV}$, the field $\vec{E}$ is normal to equipotential surfaces and directed towards decreasing potentials (without the minus sign in this relation, $\vec{E}$ is directed towards increasing potentials).We have represented in (Figure II-6) the equipotential surfaces and the lines of the electric field E created by a positive point charge. The equipotential surfaces are spheres centered at O, the point where the charge is located.

The management of $\vec{E}$, that is to say the gradient of V is the direction of the normal to the equipotential surfaces, the one where V varies the most rapidly; it is clear that to go from the value V1 to the value V2, the shortest path is the segment AB.

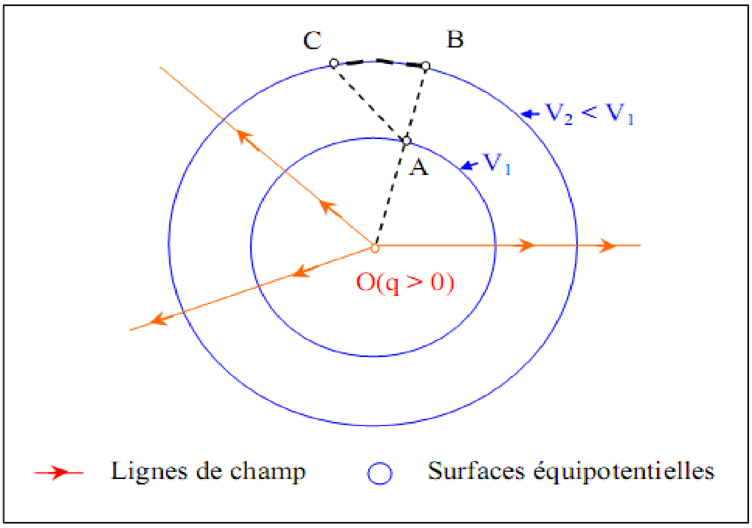

Figure 6

In the case of a system with multiple loads, the field lines cannot be obtained by superimposing the field lines of each load. The total field must be calculated. $\vec{E}$ and then draw the field lines

2.5 - Physical meaning of electrostatic potential

Let us concede a charge q at point M subjected to an electrostatic field $\vec{E}(M)$ due to a certain discontinuous charge distribution (figure 7).

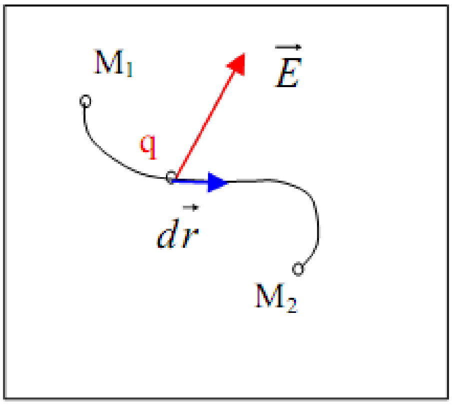

Figure 7

The electrostatic force $\vec{F} = q\vec{E}$ causes a displacement of the charge q (placed at M) from a point M1 to a point M2.

The electrostatic force is conservative. It therefore derives from a potential energy U such that:

$$dU = -dW(\vec{F}) = -\vec{F}.d\vec{r}$$

$$\text{Posons : } \vec{F}_{op} = -\vec{F}$$

We introduced the force $\vec{F}_{op}$ to have a way of bringing the charge q from point M1 to point M2, which does not result in the production of kinetic energy.

The work done to perform the quasi-static displacement of the load from M1 to M2 and to overcome the repulsive force is in the form of potential energy.

Thus, dU represents the work that an operator must apply to the charge q against the electrostatic force to move the charge q by dr.

$$dU = \vec{F}_{op}.d\vec{r} = -q\vec{E}.d\vec{r} = dW_{op} \qquad (11)$$

To move the load from point M1 to point M2, we have:

$$\Delta U = q\int_{M_1M_2}(-\vec{E}).d\vec{r} = q\int_{M_1}^{M_2} dV = q(V(M_2)-V(M_1)) = U(M_2)-U(M_1)$$

The work of force applied does not depend on the path followed; it depends only on the initial position M1 and the final position M2. It follows that the work of $\vec{F}_{op}$l when the charge q is moved along a closed contour is zero, a result we obtained for the circulation of E.

$$W_{1\to 1}^{op} = -q\oint \vec{E}.d\vec{r} = 0 \Leftrightarrow \text{conservation de l'énergie} \Leftrightarrow \text{le champ } \vec{E} \text{ est conservatif.}$$

Let's express the work $W_{\infty\to M}^{op}$that the operator must provide to bring the charge q from infinity to point M. Given that V(∞)=0:

$$W_{\infty\to M}^{op} = -\int_{\infty}^{M} q\vec{E}.d\vec{r} = qV(M) - qV(\infty) = U(M) \qquad (12)$$

U(M) is the potential energy of the charge q placed at point M where the potential is V(M), hence the name potential and the justification for the choice of the minus sign in the defining relation:

$$\vec{E} = -\overrightarrow{grad}V$$

U(M) = EP = q V(M) : potential energy of the charge q placed at a point M where the potential is equal to V(M).Potential energy is defined up to a constant. The same is true for potential. Therefore, a reference point is necessary. Experimentally, only potential differences are accessible.

3 - Continuous charge distribution – density

At the macroscopic scale, the number of elementary charges is so large that the discontinuous nature of charge no longer makes sense; the same is true for mass, since we cannot detect protons and electrons at the macroscopic scale. This allows us to consider the distribution of charges in matter as continuous.

3.1 - Linear charge density

If the charge is concentrated on a wire-like system, we define a linear charge density λ(P), based on the charge dq carried by an element dl of the wire, surrounding the point P:

$$\mathbf{dq = \lambda dl} \qquad \mathbf{(13)}$$

The total charge of the wire is given by the line integral:

$$Q = \int_{\Gamma} \lambda dl$$

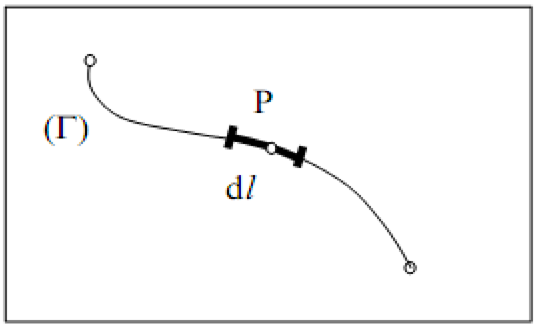

Figure 8

3.2 - Surface charge density

When the loads are distributed over a layer of very small thickness compared to the dimensions of theFor a layer, we define a surface charge density σ(P) from the charge dq carried by an element dS of the layer's surface, surrounding point P:

$$\mathbf{dq = \sigma dS \ (14)}$$

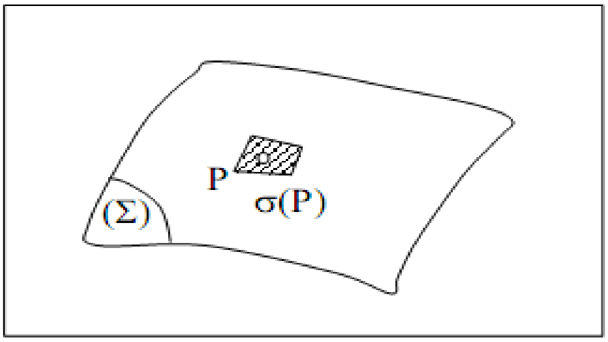

Figure 9

In this case, the total charge of a surface (S) is given by and is obtained from the surface integral:

$$Q = \iint_S \sigma dS$$

3.3 - Volumetric charge density

To describe a volume charge distribution, we define the volume charge density ρ(P) fromthe charge dq contained in a volume element dτ surrounding point P:

$$dq = \rho d\tau \quad (15)$$

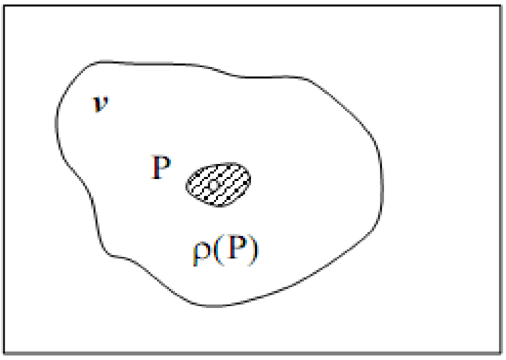

Figure 10

The charge density ρ(P) is a scalar function that can vary significantly from one point to another in the distribution. Indeed, the charge is zero in the empty space between a nucleus and an electron and takes on a non-zero value at a point located on the nucleus or the electron. Consequently, ρ(P) could have very different values depending on the choice of the elementary volume dτ. For the definition of ρ(P) to be meaningful, that is, independent of the exact form of dτ, we must consider a volume element dτ that is large compared to atomic dimensions but very small compared to the dimensions of the charge distribution. This then corresponds to a macroscopic system, and ρ(P) can be considered as a volume charge density averaged over the volume dτ. For a volume τ, the total charge is obtained from the volume integral:

$$Q = \iiint_\tau \rho d\tau$$

4 - Scope and potential of a continuous charge distribution

4.1 – Introduction

We know how to determine the electrostatic field and potential created by a distribution of point charges:

$$\vec{E}(M) = \sum_{i=1}^{n} \vec{E}_i(M) = \sum_{i=1}^{n} \frac{q_i}{4\Pi\varepsilon_0} \frac{\vec{u}_i}{r_i^2} \quad \text{et} \quad V(M) = \sum_{i=1}^{n} V_i(M) = \sum_{i=1}^{n} \frac{q_i}{4\Pi\varepsilon_0} \frac{1}{r_i}$$

The charge distribution can be divided into volume, surface, or curve elements that carry an elementary charge dq. Each of these elementary charges creates an electrostatic field and potential called elementary.

The field (or potential) created by the entire distribution is, by application of the superposition principle, the sum of the elementary charges (or potentials) created by the charges dq.

4.2 - Linear distribution

We consider a portion of a curve Γ = AB carrying a linear charge density denoted λ (figure 8).

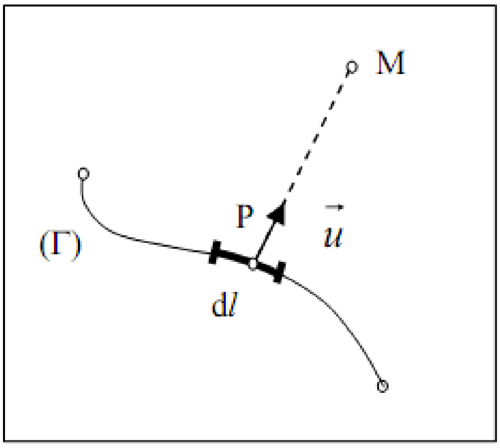

Figure 8

An element dl surrounding a point P carries a charge:

$$dq = \lambda\, dl$$

This charge creates at point M a field and a potential given by the following expressions:

$$d\vec{E}(M) = \frac{1}{4\Pi\varepsilon_0}\frac{dq}{r^2} = \frac{1}{4\Pi\varepsilon_0}\frac{\lambda(P)dl}{r^2}\vec{u} \text{ et } dV(M) = \frac{1}{4\Pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\Pi\varepsilon_0}\frac{\lambda(P)dl}{r}$$

$$\text{avec, } \overrightarrow{PM} = \left\|\overrightarrow{PM}\right\|\vec{u} = r\vec{u}$$

Hence the total field $\vec{E}(M)$ and the potential V(M) created at M by the entire linear charge distribution can be written as:

$$\vec{E}(M) = \frac{1}{4\Pi\varepsilon_0}\int_\Gamma \frac{\lambda(P)\,dl}{r^2}\vec{u} = \frac{1}{4\Pi\varepsilon_0}\int_\Gamma \frac{\lambda(P)\,dl}{\left\|\overrightarrow{PM}\right\|^3}\overrightarrow{PM} \quad (16)$$

$$V(M) = \frac{1}{4\Pi\varepsilon_0}\int_\Gamma \frac{\lambda(P)\,dl}{r} = \frac{1}{4\Pi\varepsilon_0}\int_\Gamma \frac{\lambda(P)\,dl}{\left\|\overrightarrow{PM}\right\|} \quad (17)$$

This relationship holds true for finite-dimensional wires.

4.3 - Surface distribution

In the case of a surface charge distribution, we consider a charge dq carried by a surface element dS (figure 9).

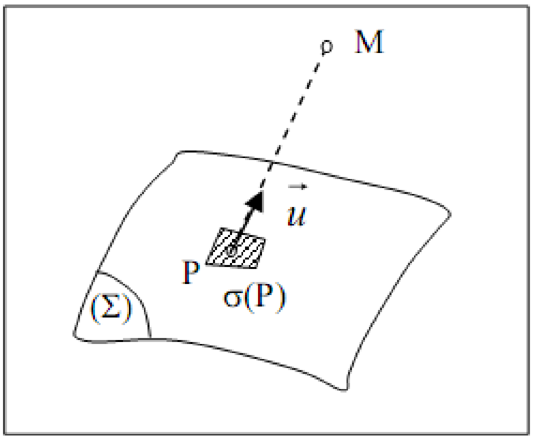

Figure 9

The field and potential created at point M by dq are given by:

$$d\vec{E}(M) = \frac{1}{4\Pi\varepsilon_0}\frac{\sigma(P)dS_P}{r^2}\vec{u} \text{ et } dV(M) = \frac{1}{4\Pi\varepsilon_0}\frac{\sigma(P)dS_P}{r}$$

$$\text{avec, } \overrightarrow{PM} = \left\|\overrightarrow{PM}\right\|\vec{u} = r\vec{u}$$

Hence the total field $\vec{E}(M)$ and the potential V(M) created by the charges distributed over the entire surface Σ

$$\vec{E}(M) = \frac{1}{4\Pi\varepsilon_0}\iint_\Sigma \frac{\sigma(P)\,dS_P}{r^2}\vec{u} = \frac{1}{4\Pi\varepsilon_0}\iint_\Sigma \frac{\sigma(P)\,dS_P}{\left\|\overrightarrow{PM}\right\|^3}\overrightarrow{PM} \qquad (18)$$

$$V(M) = \frac{1}{4\Pi\varepsilon_0}\iint_\Sigma \frac{\sigma(P)\,dS_P}{r} = \frac{1}{4\Pi\varepsilon_0}\iint_\Sigma \frac{\sigma(P)\,dS_P}{\left\|\overrightarrow{PM}\right\|} \qquad (19)$$

This relationship assumes that the charge distribution extends over a finite-dimensional surface. Otherwise, a point at a finite distance will be chosen as the origin of the potentials.

4.4 - Volume distribution

Let a volume distribution of charges contained in the volume v be given; ρ(P) is the volume charge density at a point P of the volume v (figure10).

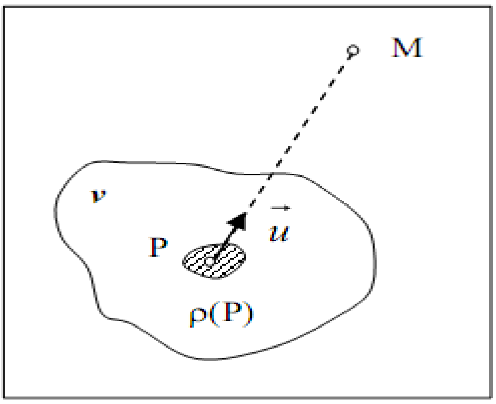

Figure 10

The charge contained in the volume element surrounding point P dτ P is:

$$dq = \rho(P)d\tau_P$$

This charge creates a field at point M $d\vec{E}$ and a potential dV as would be the case for a point charge dq placed at P (Figure 1):

$$d\vec{E}(M) = \frac{1}{4\Pi\varepsilon_0}\frac{dq}{r^2}\vec{u} \quad \text{et} \quad dV(M) = \frac{1}{4\Pi\varepsilon_0}\frac{dq}{r}$$

$$\text{avec, } \overrightarrow{PM} = \left\|\overrightarrow{PM}\right\|\vec{u} = r\vec{u} \quad \text{et} \quad dq = \rho(P)d\tau_P$$

According to the principle of superposition, the total field $\vec{E}(M)$ created by the distribution is the sum of the contributions $d\vec{E}(M)$:

$$\vec{E}(M) = \frac{1}{4\Pi\varepsilon_0}\iiint_v \frac{\rho(P)\,d\tau_P}{r^2}\vec{u} = \frac{1}{4\Pi\varepsilon_0}\iiint_v \frac{\rho(P)\,d\tau_P}{\left\|\overrightarrow{PM}\right\|^3}\overrightarrow{PM} \qquad (20)$$

Therefore, a volume integral must be calculated to obtain the field $\vec{E}(M)$ whereas the potential is obtained from the volume integral:

$$V(M) = \frac{1}{4\Pi\varepsilon_0}\iiint_v \frac{\rho(P)\,d\tau_P}{r} = \frac{1}{4\Pi\varepsilon_0}\iiint_v \frac{\rho(P)\,d\tau_P}{\left\|\overrightarrow{PM}\right\|} \qquad (21)$$

This relationship assumes that the potential at infinity is zero, meaning that the charge distribution extends over a finite volume. If this is not the case, another origin for the potentials must be chosen.

5 – Conclusion

The electrostatic field can be characterized simply using a function we will call the electrostatic potential. This scalar function is often easier to determine than the electrostatic field itself. This name is justified by the interpretation of this function in terms of the potential energy of a charge subjected to the effects of an electrostatic field.