I. The Electrostatic Dipole

I.1 Introduction

An electrostatic dipole is defined by a specific distribution of electric charges such that the center of gravity of the positive charges does not coincide with that of the negative charges (the system is overall neutral). The simplest dipole is therefore a pair of two charges of opposite sign separated by a non-zero distance "a". This concept is primarily used in electromagnetism and subsequently in chemistry, where certain bonds between molecules can be explained by modeling these molecules as a dipole (hydrogen bonds, for example). In physics, the focus is on the electrostatic field. $\vec{E}(r)$created at a point r far from the dipole (this is called an active dipole). But we can also study the behavior of the dipole when it is placed in an external field (this is called a passive dipole).

I.2 Electrostatic potential and field created by an electrostatic dipole

2.1 – Definition

An electrostatic dipole consists of two equal electric charges of opposite signs (-q) and (+q) (q > 0) (Figure 1). These two charges are fixed at two points A and B, separated by a distance $(a = \|\overrightarrow{AB}\|)$We propose to study the characteristics of the electrostatic field and potential created by these two charges at a point M very far from the charges: $a \ll r = \|\overrightarrow{OM}\|$: dipolar approximation.

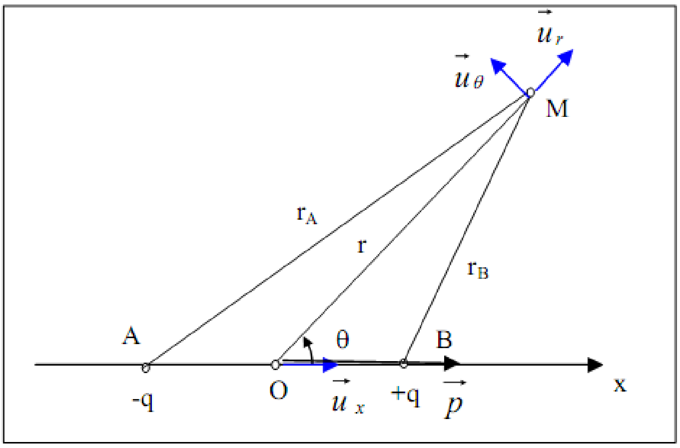

Figure 1

2.2 - Electric dipole moment

Consider two-point charges –q, +q fixed respectively at A and B (q > 0). The electric dipole moment (or dipole moment) is a vector quantity defined by (Figure 1):

$$\vec{p} = -q\overrightarrow{OA} + q\overrightarrow{OB} = q\,\overrightarrow{AB}$$

Denoting by a the distance separating A and B, the magnitude of the dipole moment is:

$$p = \left\| \vec{p} \right\| = qa$$

The dipole moment describes the charge and its geometry. It allows the dipole to be characterized. Its unit in the International System of Units (SI) is the coulomb-meter (Cm).

2.3 - Calculation of the electrostatic potential

Let two-point charges –q, +q be fixed respectively at A and B (figure 1) separated by a distance (a). Consider a point M very far from the charges, which is equivalent to considering the distance a much smaller than that which separates M from either charge. The position of M is located in the polar coordinate system (r, θ). We choose to take as axis (Ox), the line which joins the two charges such that the origin O is at the midpoint of the segment AB which joins the charges (Ox is the axis of revolution of the distribution).
According to the superposition principle, the potential V(M) created by the dipole at a point M located by its polar coordinates (r, θ) is given by:

$$V(M) = V_A(M) + V_B(M) = \frac{q_A}{4\Pi\varepsilon_0}\frac{1}{r_A} + \frac{q_B}{4\Pi\varepsilon_0}\frac{1}{r_B} = \frac{q}{4\Pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right) = \frac{q}{4\Pi\varepsilon_0}\left(r_B^{-1} - r_A^{-1}\right)$$

with,

$$* \ r_B = \left\| \overrightarrow{BM} \right\|$$

$$\overrightarrow{BM} = \overrightarrow{BO} + \overrightarrow{OM}$$

$$r_B^2 = \left\| \overrightarrow{BM} \right\|^2 = (\overrightarrow{BO} + \overrightarrow{OM})^2 = \overrightarrow{BO}^2 + 2\overrightarrow{BO}.\overrightarrow{OM} + \overrightarrow{OM}^2$$

$$\text{où, } \left\| \overrightarrow{OM} \right\| = r \ ; \ \left\| \overrightarrow{OB} \right\| = \frac{a}{2} \ \text{ et } \ \overrightarrow{BO}.\overrightarrow{OM} = \frac{a}{2} r \cos(\pi - \theta) = -\frac{ar}{2}\cos\theta$$

on a :

$$r_B^2 = \left\| \overrightarrow{BM} \right\|^2 = r^2 - ar\cos\theta + \frac{a^2}{4} = r^2 (1 - \frac{a}{r}\cos\theta + \frac{a^2}{4r^2})$$

$$* \ r_A = \left\| \overrightarrow{AM} \right\|$$

$$\overrightarrow{AM} = \overrightarrow{AO} + \overrightarrow{OM}$$

$$r_A^2 = \left\| \overrightarrow{AM} \right\|^2 = (\overrightarrow{AO} + \overrightarrow{OM})^2 = \overrightarrow{OM}^2 + 2\overrightarrow{OM}.\overrightarrow{AO} + \overrightarrow{OA}^2$$

$$\text{où, } \overrightarrow{OM}.\overrightarrow{AO} = \frac{ar}{2}\cos\theta \ \text{ et } \ \left\| \overrightarrow{OA} \right\| = \frac{a}{2}$$

So,

$$r_A^2 = \left\| \overrightarrow{AM} \right\|^2 = r^2 + ar\cos\theta + \frac{a^2}{4} = r^2 (1 + \frac{a}{r}\cos\theta + \frac{a^2}{4r^2})$$

So, we have,

$$r_A = \left\|\overrightarrow{AM}\right\| = r\left(1+\frac{a}{r}\cos\theta+\frac{a^2}{4r^2}\right)^{1/2} \text{ et } r_A^{-1} = r^{-1}\left(1+\frac{a}{r}\cos\theta+\frac{a^2}{4r^2}\right)^{-1/2}$$

$$r_B = \left\|\overrightarrow{BM}\right\| = r\left(1-\frac{a}{r}\cos\theta+\frac{a^2}{4r^2}\right)^{1/2} \text{ et } r_B^{-1} = r^{-1}\left(1-\frac{a}{r}\cos\theta+\frac{a^2}{4r^2}\right)^{-1/2}$$

Since $a/r \ll 1$, we have: $a^2/(4r^2) \ll a/r$, we can neglect the $(a/r)^2$ terms compared to the (a/r) term:

$$r_A^{-1} \cong r^{-1}\left(1+\frac{a}{r}\cos\theta\right)^{-1/2}$$

$$r_B^{-1} \cong r^{-1}\left(1-\frac{a}{r}\cos\theta\right)^{-1/2}$$

Given that $a \ll r$, we can expand r_A^{-1} et r_B^{-1} in the power of (a/r) and retain only the first-order term $(1+x)^{-1/2} = 1-\frac{1}{2}x+\ldots$

$$r_A^{-1} \cong r^{-1}\left(1-\frac{1}{2}\frac{a}{r}\cos\theta\right)$$

$$\text{et } r_B^{-1} \cong r^{-1}\left(1+\frac{1}{2}\frac{a}{r}\cos\theta\right)$$

d'où :

$$r_B^{-1} - r_A^{-1} = r^{-1}\left(1+\frac{1}{2}\frac{a}{r}\cos\theta\right) - r^{-1}\left(1-\frac{1}{2}\frac{a}{r}\cos\theta\right) = \frac{a}{r^2}\cos\theta$$

The potential V(M) is therefore given by:

$$V(M) = \frac{qa\cos\theta}{4\Pi\varepsilon_0 r^2} = \frac{p\cos\theta}{4\Pi\varepsilon_0 r^2}$$

Either $\vec{r} = \overrightarrow{OM}$ the position vector of point M relative to point O (midpoint of [A, B]) and $\vec{p}$ the dipole moment (figure 2).

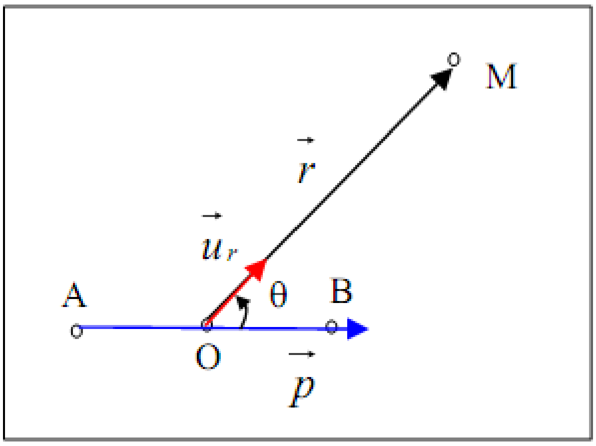

Figure 2

We have:

$$\vec{p}.\vec{r} = pr\cos\theta$$

The potential V(M) can therefore be written as:

$$\boxed{V(M) = \frac{\vec{p}.\vec{r}}{4\Pi\varepsilon_0 r^3} = \frac{\vec{p}.\vec{u}_r}{4\Pi\varepsilon_0 r^2} \qquad (3)}$$

This expression, which involves a scalar product, is independent of any coordinate system. It should be noted that the decay of the potential created by a dipole ($1/r^2$) is faster than in the case of a point charge which is in (1/r).

2.4 - Calculation of the electrostatic field

2.4.1 - Field components in polar coordinates

The dipole exhibits rotational symmetry around (AB). The electrostatic field $\vec{E}(M)$ is therefore contained in the plane (M, AB) (figure 3).

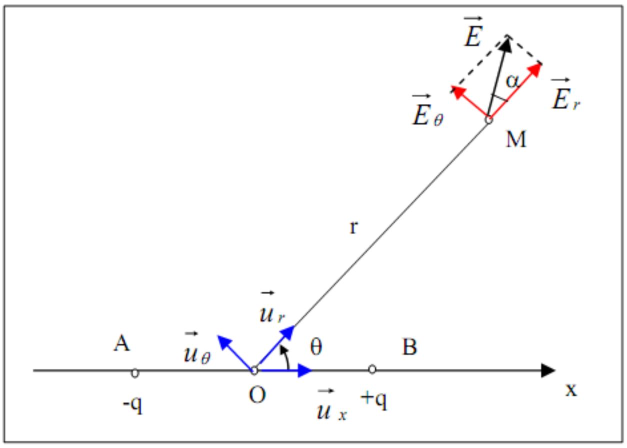

Figure 3

According to the superposition principle, the field at M is given by:

$$\vec{E}(M) = \vec{E}_A(M) + \vec{E}_B(M) = E_r\vec{u}_r + E_\theta\vec{u}_\theta \quad (\vec{E}_z = \vec{0})$$

To calculate the components of the field, we use the following relationship:

$$\vec{E}(M) = -\overrightarrow{grad}V(M)$$

$$\text{avec, } \overrightarrow{grad}V(M) = \frac{\partial V}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{u}_\theta \text{ et } V(M) = \frac{p\cos\theta}{4\Pi\varepsilon_0 r^2}$$

The components of the field derived from the potential V(M) are written in the cylindrical coordinate system:

$$\vec{E}_r = -\frac{\partial V}{\partial r}\vec{u}_r = \frac{2p\cos\theta}{4\Pi\varepsilon_0 r^3}\vec{u}_r$$

$$\vec{E}_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta}\vec{u}_\theta = \frac{2p\sin\theta}{4\Pi\varepsilon_0 r^3}\vec{u}_\theta$$

θ	0	Π/2	Π	3Π/2
$\vec{E}_r$	$\vec{E}_1 = \frac{2p}{4\Pi\varepsilon_0}\frac{\vec{u}_r}{r^3}$	$\vec{E}_r = \vec{0}$	$\vec{E}_3 = \vec{E}_r = \vec{E}_1$	$\vec{E}_r = \vec{0}$
$\vec{E}_\theta$	$\vec{E}_\theta = \vec{0}$	$\vec{E}_2 = \frac{p}{4\Pi\varepsilon_0}\frac{\vec{u}_\theta}{r^3}$ $E_2 = E_1/2$	$\vec{E}_\theta = \vec{0}$	$\vec{E}_4 = \vec{E}_2$

It should be noted that the decay of the field in (1/r^3) created by a dipole is faster than in the case of a point charge which is in (1/r²).

The module of $\vec{E}(M)$ East :

$$\boxed{\left\|\vec{E}\right\| = \frac{p}{4\Pi\varepsilon_0 r^3}\sqrt{1+3\cos^2\theta}}$$

Let α be the angle that the electric field makes with the radial: $\alpha = (\vec{E}, \vec{u}_r)$

$$tg\alpha = \frac{E_\theta}{E_r} = \frac{tg\theta}{2}$$

Note that the Cartesian components of the following field Ox and Oy (of the AMB plane) are written:

$$\vec{u}_r = \cos\theta\vec{i} + \sin\theta\vec{j} \text{ et } \vec{u}_\theta = -\sin\theta\vec{i} + \cos\theta\vec{j}$$

$$\vec{E} = \vec{E}_r + \vec{E}_\theta = \frac{2p\cos\theta}{4\Pi\varepsilon_0 r^3}(\cos\theta\vec{i} + \sin\theta\vec{j}) + \frac{p\sin\theta}{4\Pi\varepsilon_0 r^3}(-\sin\theta\vec{i} + \cos\theta\vec{j})$$

$$\vec{E} = \vec{E}_x + \vec{E}_y = \frac{p}{4\Pi\varepsilon_0 r^3}(3\cos^2\theta - 1)\vec{i} + \frac{p}{4\Pi\varepsilon_0 r^3}(3\sin\theta\cos\theta)\vec{j}$$

2.4.2 - Global formulation of the field $\vec{E}$

We can express $\vec{E}$ only depending on $\vec{p}$ and $\vec{r}$ by calculating the gradient of V(M):

$$\vec{E}(M) = -\overrightarrow{grad}V(M) = -\overrightarrow{grad}\left(\frac{\vec{p}.\vec{r}}{4\Pi\varepsilon_0 r^3}\right) = -\frac{1}{4\Pi\varepsilon_0 r^3}\overrightarrow{grad}(\vec{p}.\vec{r}) - \frac{\vec{p}.\vec{r}}{4\Pi\varepsilon_0}\overrightarrow{grad}\left(\frac{1}{r^3}\right)$$

Or,

En posant : $\vec{p} = p_x \vec{i} + p_y \vec{j} + p_z \vec{k}$ et $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$* \quad \overrightarrow{grad}(\vec{p}.\vec{r}) = \overrightarrow{grad}(p_x\, x + p_y\, y + p_z\, z) = p_x\, \vec{i} + p_y\, \vec{j} + p_z\, \vec{k} = \vec{p}$$

$$* \quad \overrightarrow{grad}\left(\frac{1}{r^3}\right) = -\frac{3}{r^5}\vec{r}$$

Hence the intrinsic expression of $\vec{E}$ depending on $\vec{p}$ and $\vec{r}$:

$$\boxed{\vec{E}(M) = \frac{1}{4\Pi\varepsilon_0}\left(\frac{3(\vec{p}.\vec{r})\,\vec{r}}{r^5} - \frac{\vec{p}}{r^3}\right) \qquad (5)}$$

Electrical effects $\vec{E}$ and V produced by the dipole are entirely determined by its dipole moment $\vec{p}$ It should be noted that the decay of the potential in (1/r²) and of the field in (1/r^3) created by a dipole is faster than in the case of a point charge.

Note that the Cartesian components of the following field Ox and Oy (of the AMB plane) can also be obtained by writing:

$$\vec{p}.\vec{r} = pr\cos\theta \ ; \ \vec{r} = r\cos\theta\vec{i} + r\sin\theta\vec{j} \ \text{ et } \ \vec{p} = p\vec{i}$$

This gives, according to the intrinsic expression of the field independent of the coordinate system:

$$\vec{E} = \vec{E}_x + \vec{E}_y = \frac{1}{4\Pi\varepsilon_0}\left(\frac{3pr\cos\theta}{r^5}(r\cos\theta\vec{i} + r\sin\theta\vec{j}) - \frac{p}{r^3}\vec{i})\right)$$

We therefore find the components calculated from the polar components of the field:

$$\vec{E}_x = \frac{p}{4\Pi\varepsilon_0 r^3}(3\cos^2\theta - 1)\vec{i} \ \text{ et } \ \vec{E}_y = \frac{p}{4\Pi\varepsilon_0 r^3}(3\sin\theta\cos\theta)\vec{j}$$