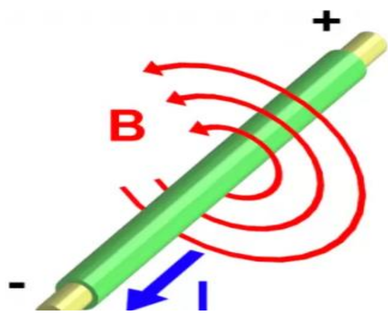
Chapter 2: Magnetostatics

2.1 Ampère's Law

Ampère's law stipulates:

- The circulation of magnetic the constant currents along any closed circuit are proportional to the sum of the forces of the currents that cross the surface of the circuit.
- If the direct is used, the magnetic field is continuous.
- If an alternating is used, the magnetic field is alternating.



Ampère's law can be represented by the following equation:

$$\oint \vec{B}\, d\vec{l} = \mu_0 I_T$$

In this formula for calculating the magnetic field, the integral represents the circulation of the field lines along a closed path, and:

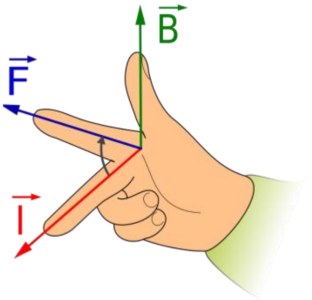
- μ0 is the permeability of free space

- dl is a tangent vector to the chosen path at each point

- IT is the net intensity of the current that passes through the surface bounded by the path, and it will be positive or negative depending on the direction in which it passes through the surface.

2.2 Direction of the magnetic field (right-hand rule)

The right-hand rule is a way to remember how various directions are related. It uses the fingers of the hand. There are two (best-known) rules: one that imitates a corkscrew with rotation and translation, and one that indicates a direct reference point.electromagnetism,These rules allow us to determine the direction and sense of Laplace(electrical machines...)and to give the shape and direction of the lines of field produced by an electric, among other things.



2.3 Magnetic Potential

The vector potential of the magnetic field, or, more simply, vector potential when there is no possible confusion, is a physical quantity that can be considered equivalent to a field of vectors intervening in electromagnetism. It is not directly measurable, but its presence is intimately linked to that of a field. Its SI unit is the Kg.C-1.ms-1

In electrostatics, the electrostatic potential at a point M is written as:

$$V(M) = \int_{\mathfrak{D}} \frac{1}{4\pi\varepsilon_0} \frac{\mathrm{d}q(P)}{PM}$$

P being a point in the load distribution $\mathfrak{D}$ And for a volumetric distribution: $\mathrm{d}q(P) = \rho(P)\,\mathrm{d}\mathfrak{V}$ [$\rho(P)$ volume charge density in P], for a surface distribution: $\mathrm{d}q(P) = \sigma(P)\,\mathrm{d}\mathfrak{S}$ [$\sigma(P)$ surface charge density at P], for a linear distribution: $\mathrm{d}q(P) = \lambda(P)\,\mathrm{d}l$ [$\lambda(P)$ linear charge density in P].

By analogy, the magnetostatic potential at a point M can be written as:

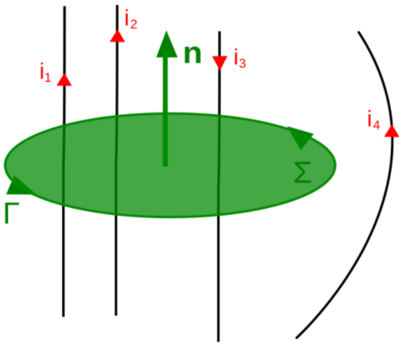
$$\vec{A}(M) = \int_{\mathfrak{D}} \frac{\mu_0}{4\pi} \frac{\overrightarrow{d\mathcal{C}}(P)}{PM},$$

Pbeing a point in the current distribution $\mathfrak{D}$ and for a volumetric current: $\overrightarrow{d\mathcal{C}}(P) = \vec{J}(P)\, d\mathfrak{V}$ [$\vec{J}(P)$ volume current at point P], for a surface current: $\overrightarrow{d\mathcal{C}}(P) = \vec{J}_s(P)\, d\mathfrak{S}_s$ [$\vec{J}_s(P)$ surface current at point P], for a linear current: $\overrightarrow{d\mathcal{C}}(P) = I\, \overrightarrow{dl}$ (I current intensity).

2.4 Ampère's Theorem

The circulation of magnetic excitation $\vec{H}$ along a closed curve C is equal to the total intensity that passes through any surface resting on C This assumes, of course, that we are in steady state, in which case the current density vector $\vec{J}$ is at conservative flux and the intensity depends only on C and not the choice of surface based on C.

$$\int_C \vec{H}.\overrightarrow{dl} = \int\int_S \vec{J}.\overrightarrow{dS}$$



Ampère's theorem allows us to determine the field created by current elements. Ampère's theorem is the analogue of Gauss's theorem in electrostatics. The magnetic field B created by a current I is given by Ampère's theorem:

$$\oint_\Gamma \vec{B} \cdot d\vec{\ell} = \mu_0 I$$

where Γ is any closed curve through which the electric current flowsI.

$\mu 0 = 4\pi \times 10 - 7$ H m-1

2.5 Maxwell-Ampère Theorem

Ampère's theorem can be generalized by applying it to the total current. The Maxwell-Ampère relation, which translates it, is written:

$$\oint_{(\Gamma)} \vec{B} \cdot d\vec{\ell} = \iint_{(S)} \mu_0 \vec{j}_T \cdot d\vec{S} = \iint_{(S)} \left(\mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{S} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

The integral relation of the generalized Ampère's theorem is:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}_T = \mu_0 \vec{j} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

The fundamental result is obtained: A changing electric field creates a magnetic field.

2.6 Magnetic Flux

2.6.1 Elementary Flux

Consider an elementary surface dS in the neighborhood of point M. Let ~n be a normal vector to this surface. The choice of the direction of ~n defines the orientation of the surface. We then define $d\vec{S} = dS\,\vec{n}$ the elementary surface vector. The magnetic field flux through the oriented surface dS is defined by:

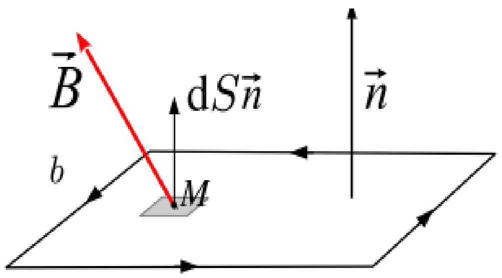
$$d\phi = \vec{B}(M).d\vec{S}$$

-The unit of heat flux is the weber (Wb) 1 W = 1 T.m2

-The sign of dφ depends on the chosen direction of orientation.

2.6.2 Flux of a uniform magnetic field through a loop

Consider a rectangular loop, with sides a and b, placed in a uniform magnetic field. The direction of ~n is deduced from the orientation chosen for the loop by the right-hand rule.



By definition, the total flux of B~ through the oriented surface delimited by the loop is

$$\phi = \iint \vec{B}(M).\mathrm{d}\vec{S} = \iint \vec{B}(M).\mathrm{d}S\,\vec{n}$$

Since the magnetic field B is uniform, it can be removed from the integral. Furthermore, since the surface is flat, all the vectors' dS are collinear with the same vector n:

$$\phi = \vec{B}.\iint \mathrm{d}S\,\vec{n} = \vec{B}.\left[\iint \mathrm{d}S\right]\vec{n} = \vec{B}.S\vec{n}$$

where S = ab represents the area of the surface delimited by the spiral.

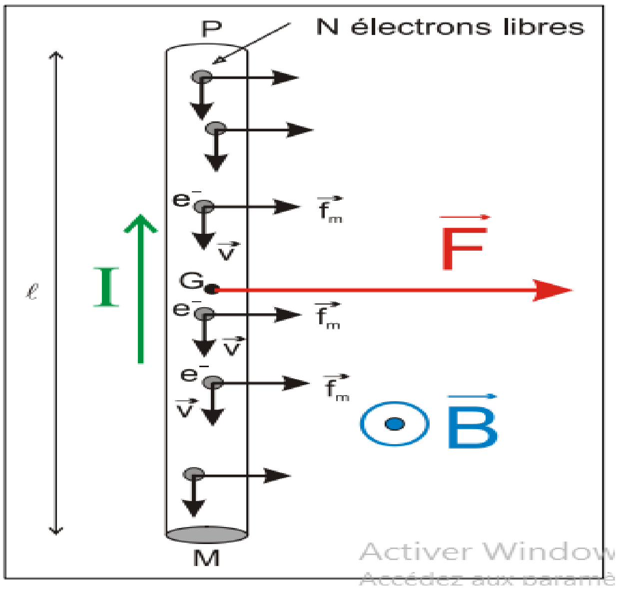
The magnetic flux through any closed surface S is zero. We say that the magnetostatic field has conservative flux. This property is expressed by the following integral:

2.7 Laplace force

Consider a straight conductor of length l = PM carrying an electric current of intensity I and placed in a magnetic field B perpendicular to PM. The N free electrons contained in this conductor, constituting the current, with charge q = -e, move with a certain velocity through B. They therefore all experience a Lorentz force:

$$f_m = |qvB\sin\alpha| = evB\sin\alpha$$

The resultant of the N Lorentz forces constitutes the electromagnetic Laplace force exerted on the entire conductor.



In order to determine the force, we reason on the simplified model of electric current where the N free electrons move at the same constant speed v.

Under these conditions, the N electrons experience the same Lorentz force. The Laplace force is given by:

$$F = Nf_m = N\left|qvB\sin\alpha\right| = NevB\sin\alpha \qquad \text{avec } \alpha = \text{angle entre } q\vec{v} \text{ et } \vec{B}.$$

2.8 Magnetic Energy

2.8.1 Definition

Let us first consider the case of a conducting bar MN moving on parallel conducting rails. The system is powered by a direct current source that delivers a current I through the circuit thus formed. This circuit is placed in a uniform magnetic field Bρ with normalθperpendicular to the bar MN and making an angleθ with the normal to the circuit plane.

The conductor MN is subject to the Laplace force:

$$\vec{F} = I\ \overline{MN} \wedge \vec{B} \quad \Rightarrow \quad \left|\vec{F}\right| = I \left|\overline{MN}\right| \left|\vec{B}\right| \quad \text{soit} \quad F = I\,MN\,B$$

During an elementary displacement dM, the work done by the force is:

$$dW = \vec{F}\,.\,\overrightarrow{dM} \quad \text{soit} \quad dW = F\ dM\ \cos\theta$$

By replacing force with its expression, he deduces that:

$$dW = I\ MN\ dM\ \ B\cos\theta \qquad \text{soit} \qquad dW = I\ dS\ \ B\cos\theta$$

With: dS is the surface swept by the conductor MN during this movement.

If we consider the flow interrupted by the conductor:

$$d\Phi = B\,dS\,\cos\theta$$

The following results are obtained:

$$dW = I\ d\Phi$$

2.8.2Electromagnetic field energy.

An electromagnetic field contains and transports energy. We define:

- **Volumetric electromagnetic energy.**

The homogeneous quantity described by the equation for a volumetric energy (expressed in SI units in J/m3) is called volumetric electromagnetic energy (or improperly volumetric electromagnetic energy density).

$$\varpi = \frac{1}{2}\varepsilon_0 \left\|\vec{E}\right\|^2 + \frac{\left\|\vec{B}\right\|^2}{2\mu_0}$$

This expression shows that the energy is localized within the electromagnetic field itself. By isolating the contributions due to the fields EA Nd B We can distinguish:

$$\varpi_{él} = \frac{1}{2}\varepsilon_0 \left\|\vec{E}\right\|^2 \text{, } \underline{\textit{énergie électrique volumique}} \quad \text{et} \quad \varpi_{mag} = \frac{\left\|\vec{B}\right\|^2}{2\mu_0} \text{, } \underline{\textit{énergie magnétique volumique}}.$$

- **The Poynting vector**

On note $\vec{\Pi}$ le vecteur, appelé ***vecteur de Poynting***, défini par : $\vec{\Pi} = \frac{\vec{E} \times \vec{B}}{\mu_0}$

$\left\|\vec{\Pi}\right\|$ est homogène à une **puissance surfacique**, exprimé en SI en $W.m^{-2}$

In summary;

In summary, here is some advice on the methods to use to calculate the field
magnetic.

- **The Biot and Savart formula:** It is only practical if you know how to do addition.
vector of dB fields created by a small element of the circuit (often circuits
threadlike).
- **Conservation of flow:** to be used only if its expression is already known in another region of space.

- **Ampère's theorem:** It is necessary to be able to calculate the circulation of the field along a chosen contour. This therefore requires a relatively simple symmetry of the currents.

Exercise 1:Magnetic field created by a loop

Using the Biot-Savart formula, determine the characteristics of the magnetic field created at the center of a flat coil of N turns, of radius R and carrying a current I. Numerical application: R = 5 cm, N = 100 and I = 100 mA.

Exercise 2:Magnetic field created by a cable

Consider a cable of radius R and infinite length, carrying a current of intensity I uniformly distributed across its cross-section. Using Ampère's law, determine the magnetic field strength at a point located at a distance r from the cable's axis. Plot the curve B(r).

Exercise 3:Magnetic field created by a coaxial cable

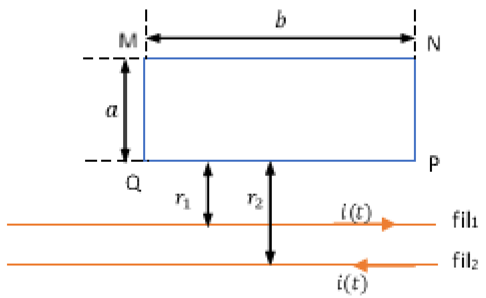
Consider an infinite cylindrical coaxial cable with radii R1, R2 and R3. The current of total intensity I passes in one direction in the inner conductor and returns in the other direction through the outer conductor.

Exercise 4

Two parallel, thread-like conductors, 1 and 2, carrying the same current of intensity $i(t) = I_0\cos \omega t$, in opposite directions. A rectangular frame MNPQ set fixed in the plane of the conductors.

1- Determine the magnetic fluxφthrough the frame.

2- Determine the induced electromotive force$e(t)$within the framework and to give the conventional direction of the induced currenti'.

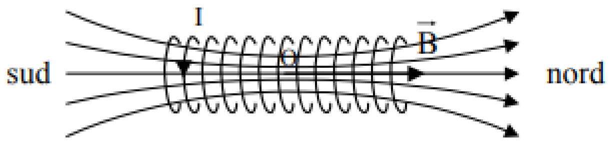


Exercise 1

a/ The magnetic spectrum of a solenoid is similar to that of a bar magnet.

The field lines are oriented using the right-hand rule (the direction of the current must first be defined). From this, the north and south faces of the solenoid are deduced.

The magnetic field at the center of the solenoid is tangent to the field line passing through O and has a direction given by the orientation of the field line.

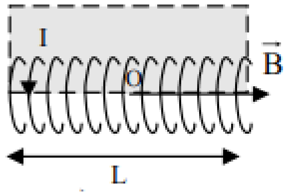


b/ We assume that inside the solenoid the field is uniform and outside it is zero.

The circulation of the magnetic field along the contour (C) is: C = BL (see figure). Applying Ampère's theorem gives: C = Nμ0I

D'où : $B = \mu_0 \dfrac{N}{L} I$

A.N. $B = 3{,}1 \cdot 10^{-5}$ T



c) The needle points towards magnetic north (Earth's magnetic field).

$$dB = \frac{\mu_0 I}{4\pi}\frac{d\ell\, R}{R^3} = \frac{\mu_0 I}{4\pi}\frac{d\ell}{R^2}$$

Au totale, la longueur de la bobine est N2πR.

$$B = \frac{\mu_0 I}{4\pi}\frac{N2\pi R}{R^2} = N\frac{\mu_0 I}{2R}$$

A.N. B = 0,126 mT

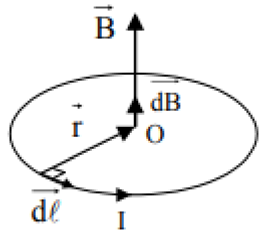
Exercise 2

A piece of coil of length dl makes the following contribution:

Un morceau de bobine de longueur $d\ell$ apporte la contribution : $d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{\ell} \wedge \vec{r}}{r^3}$

Ce champ élémentaire est dirigé suivant l'axe et son sens dépend du sens du courant (voir figure).

$dB = \frac{\mu_0 I}{4\pi} \frac{d\ell R}{R^3} = \frac{\mu_0 I}{4\pi} \frac{d\ell}{R^2}$



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Le sens du champ magnétique s'obtient avec la règle de la main droite.

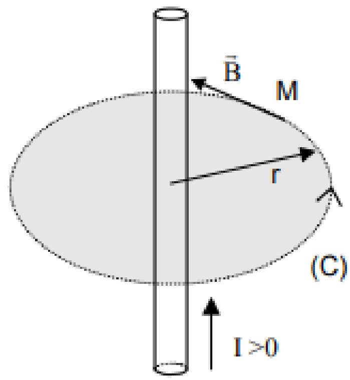
- Champ magnétique à l'extérieur du câble (r >R) :

Appliquons le théorème d'Ampère avec un contour circulaire (C) centré sur le câble.

La circulation s'écrit : $C = B\ 2\pi r$

Théorème d'Ampère : $C = \mu_0 I$

D'où : $B = \frac{\mu_0 I}{2\pi r}$

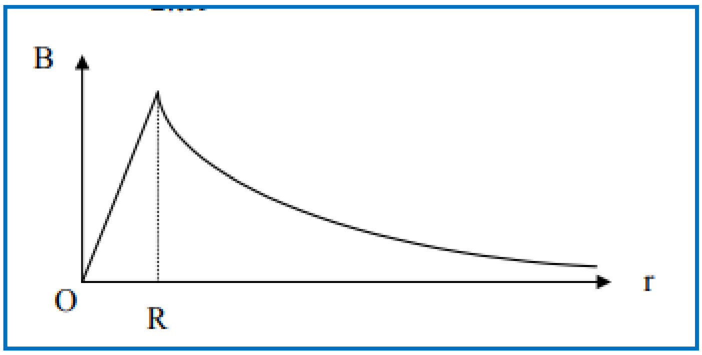


- Champ magnétique à l'intérieur du câble (r ≤ R) :

Dans la section de rayon r passe le courant : $J = I\frac{\pi r^2}{S} = I\frac{r^2}{R^2}$

$C = B\ 2\pi r = \mu_0 J$

D'où : $B = \frac{\mu_0 I}{2\pi R^2} r$



Exercise 4

1- • On peut alors calculer le flux Φ du champ magnétique $\vec{B}$ à travers le cadre. D'après le sens de la normale, le cadre est orienté de c vers A.

Le flux à travers chaque spire du cadre est identique. Donc le flux à travers le cadre est égal à N fois le flux à travers une spire.

On a $\Phi = N \iint_S \vec{B} \cdot d\vec{S}$ où S représente la surface du cadre et $d\vec{S} = dS\vec{n} = dS(cos\theta\vec{u}_x + sin\theta\vec{u}_y)$.

Soit : $\boldsymbol{\Phi = NB_o a^2 sin\theta cos\omega t}$.

• D'après la loi de Faraday, $e = -\frac{d\Phi}{dt}$ lorsque la force électromotrice e est orienté de C vers A.

On en déduit : $\boldsymbol{e(t) = NB_o a^2 \omega sin\theta cos\omega t}$.

2- On calcule le potentiel vecteur $\vec{A}$ proposé, en un point $M(x, y, z)$ quelconque dans la base cartésienne.

$$\vec{A} = \frac{\vec{B} \wedge \overrightarrow{OM}}{2} = \frac{1}{2}\begin{pmatrix} 0 \\ B \\ 0 \end{pmatrix} \wedge \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} zB \\ 0 \\ -xB \end{pmatrix}$$

On vérifie qu'il est bien potentiel vecteur du champ magnétique $\vec{B}$.

$$\overrightarrow{rot}\vec{A} = \vec{\nabla} \wedge \vec{A} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \wedge \begin{pmatrix} \frac{zB}{2} \\ 0 \\ -\frac{xB}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{B}{2} + \frac{B}{2} \\ 0 \end{pmatrix} = \vec{B}$$

$$e(t) = \int_C^A \vec{E}_m \cdot d\vec{l} = \int_C^F \vec{E}_m \cdot d\vec{l} + \int_F^E \vec{E}_m \cdot d\vec{l} + \int_E^D \vec{E}_m \cdot d\vec{l} + \int_D^A \vec{E}_m \cdot d\vec{l}$$

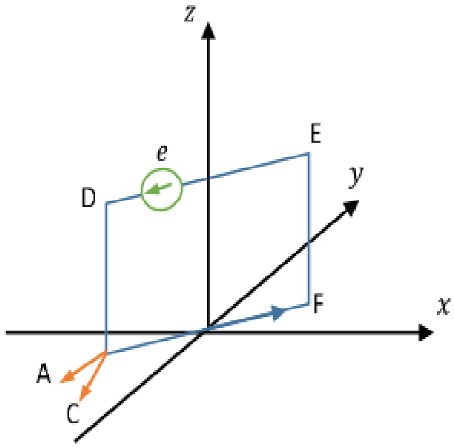
Sur FE et DA, $d\vec{l} = dz\,\vec{u}_z$ et donc $\vec{E}_m \cdot d\vec{l} = \frac{x}{2}\frac{\partial B}{\partial t}dz$

$$\int_F^E \vec{E}_m \cdot d\vec{l} = \int_0^a \frac{-\frac{a\sin\theta}{2}}{2}\frac{\partial B}{\partial t}dz = -\frac{a^2\sin\theta}{4}\frac{\partial B}{\partial t} = \frac{a^2\sin\theta}{4}B_o\omega\sin\omega t$$

3- Le champ électromoteur $\vec{E}_m$ est défini par :

$$\vec{E}_m = -\frac{\partial\vec{A}}{\partial t} = \begin{pmatrix} -\frac{z}{2}\frac{\partial B}{\partial t} \\ 0 \\ \frac{x}{2}\frac{\partial B}{\partial t} \end{pmatrix}, \text{ dans la base cartésienne}$$

On calcule la circulation de ce champ électromoteur entre les deux points C et A ($d\vec{l}$ va de C vers A) , et on obtient la force électromotrice d'induction e.



$$\int_D^A \vec{E}_m \cdot d\vec{l} = \int_a^0 \frac{\frac{a\sin\theta}{2}}{2}\frac{\partial B}{\partial t}dz = -\frac{a^2\sin\theta}{4}\frac{\partial B}{\partial t} = \frac{a^2\sin\theta}{4}B_o\omega\sin\omega t$$

Sur CF et ED, $d\vec{l} = dx\,\vec{u}_x + dy\vec{u}_y$ et donc $\vec{E}_m \cdot d\vec{l} = -\frac{z}{2}\frac{\partial B}{\partial t}dx$, soit :

$$\int_C^F \vec{E}_m \cdot d\vec{l} = \int_{\frac{a\sin\theta}{2}}^{-\frac{a\sin\theta}{2}} 0\,dx = 0 \text{ car le segment [CF] est à une altitude z } = 0.$$

$$\int_E^D \vec{E}_m \cdot d\vec{l} = \int_{\frac{a\sin\theta}{2}}^{-\frac{a\sin\theta}{2}} -\frac{a}{2}\frac{\partial B}{\partial t}dx = \frac{a^2\sin\theta}{4}B_o\omega\sin\omega t$$

On obtient donc :

$$\boldsymbol{e(t) = NB_o a^2 \omega \sin\theta \cos\omega t}$$

4- L'amplitude de la force électromotrice est une grandeur toujours positive.

Donc $\mathbf{e(t) = NB_o a^2 \omega |\sin\theta|}$

Application numérique : $\boldsymbol{e_\theta = 942\ mV}$.