DTFT Properties

- Example - Determine the DTFT $Y(e^{j\omega})$ of

$$y[n] = (n+1)\alpha^n \mu[n], \ |\alpha| < 1$$

- Let $x[n] = \alpha^n \mu[n], |\alpha| < 1$
- We can therefore write

$$y[n] = n\,x[n] + x[n]$$

- From Table 3.1, the DTFT of $x[n]$ is given by

$$X(e^{j\omega}) = \frac{1}{1 - \alpha\, e^{-j\omega}}$$

DTFT Properties

- Using the differentiation property of the DTFT given in Table 3.2, we observe that the DTFT of $nx[n]$ is given by

$$j\frac{dX(e^{j\omega})}{d\omega} = j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

- Next using the linearity property of the DTFT given in Table 3.2 we arrive at

$$Y(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} + \frac{1}{1-\alpha e^{-j\omega}} = \frac{1}{(1-\alpha e^{-j\omega})^2}$$

DTFT Properties

- Example - Determine the DTFT $V(e^{j\omega})$ of the sequence $v[n]$ defined by

$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$$

- From Table 3.1, the DTFT of $\delta[n]$ is 1
- Using the time-shifting property of the DTFT given in Table 3.2 we observe that the DTFT of $\delta[n-1]$ is $e^{-j\omega}$ and the DTFT of $v[n-1]$ is $e^{-j\omega}V(e^{j\omega})$

DTFT Properties

- Using the linearity property of Table 3.2 we then obtain the frequency-domain representation of

$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$$

as

$$d_0 V(e^{j\omega}) + d_1 e^{-j\omega} V(e^{j\omega}) = p_0 + p_1 e^{-j\omega}$$

- Solving the above equation we get

$$V(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega}}{d_0 + d_1 e^{-j\omega}}$$

Energy Density Spectrum

- The total energy of a finite-energy sequence $g[n]$ is given by

$$\mathsf{E}_g = \sum_{n=-\infty}^{\infty} \left| g[n] \right|^2$$

- From Parseval's relation given in Table 3.2 we observe that

$$\mathsf{E}_g = \sum_{n=-\infty}^{\infty} \left| g[n] \right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| G(e^{j\omega}) \right|^2 d\omega$$

Energy Density Spectrum

- The quantity

$$S_{gg}(\omega) = \left| G(e^{j\omega}) \right|^2$$

is called the **energy density spectrum**

- The area under this curve in the range $-\pi \leq \omega \leq \pi$ divided by 2π is the energy of the sequence

Energy Density Spectrum

- Example - Compute the energy of the sequence

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

- Here

$$\sum_{n=-\infty}^{\infty} \left| h_{LP}[n] \right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| H_{LP}(e^{j\omega}) \right|^2 d\omega$$

where

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

Energy Density Spectrum

- Therefore

$$\sum_{n=-\infty}^{\infty} \left| h_{LP}[n] \right|^2 = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi} < \infty$$

- Hence, $h_{LP}[n]$ is a finite-energy sequence

DTFT Computation Using MATLAB

- The function `freqz` can be used to compute the values of the DTFT of a sequence, described as a rational function in in the form of

$$X(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega} + \ldots. + p_M e^{-j\omega M}}{d_0 + d_1 e^{-j\omega} + \ldots. + d_N e^{-j\omega N}}$$

at a prescribed set of discrete frequency points $\omega = \omega_\ell$

DTFT Computation Using MATLAB

- For example, the statement

  `H = freqz(num,den,w)`

  returns the frequency response values as a vector `H` of a DTFT defined in terms of the vectors `num` and `den` containing the coefficients $\{p_i\}$ and $\{d_i\}$, respectively at a prescribed set of frequencies between `0` and 2π given by the vector `w`

DTFT Computation Using MATLAB

- There are several other forms of the function `freqz`
- The Program 3_1 in the text can be used to compute the values of the DTFT of a real sequence
- It computes the real and imaginary parts, and the magnitude and phase of the DTFT

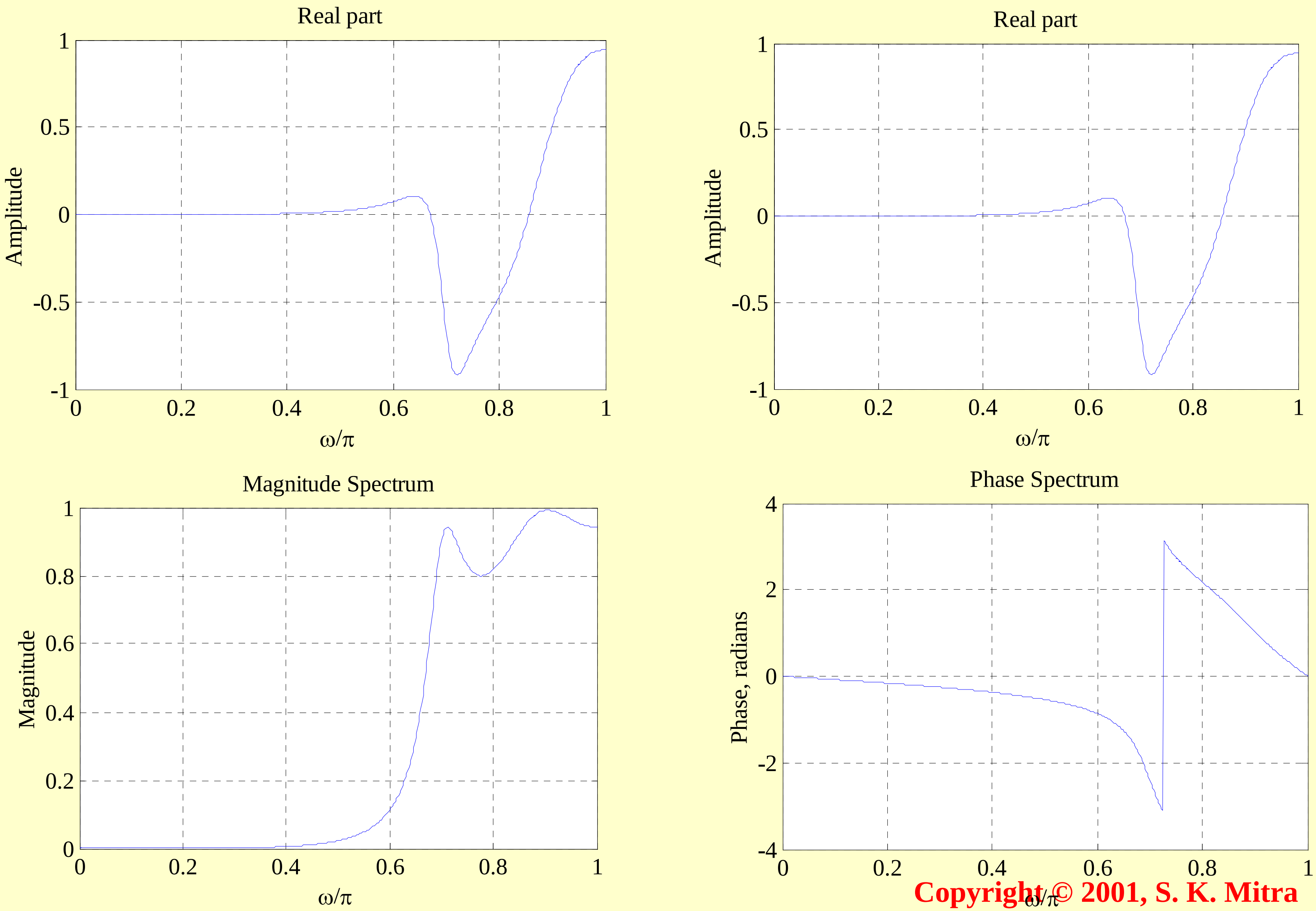
DTFT Computation Using MATLAB

- <u>Example</u> - Plots of the real and imaginary parts, and the magnitude and phase of the DTFT

$$X(e^{j\omega}) = \frac{\begin{array}{c}0.008 - 0.033e^{-j\omega} + 0.05e^{-j2\omega} \\ -0.033e^{-j3\omega} + 0.008e^{-j4\omega}\end{array}}{\begin{array}{c}1 + 2.37e^{-j\omega} + 2.7e^{-j2\omega} \\ +1.6e^{-j3\omega} + 0.41e^{-j4\omega}\end{array}}$$

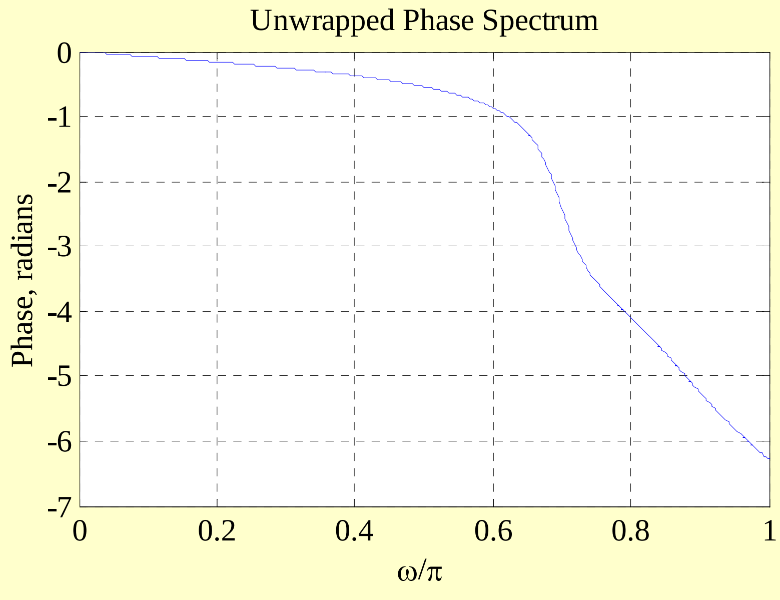
are shown on the next slide

DTFT Computation Using MATLAB



DTFT Computation Using MATLAB

- Note: The phase spectrum displays a discontinuity of 2π at $\omega = 0.72$
- This discontinuity can be removed using the function `unwrap` as indicated below



Linear Convolution Using DTFT

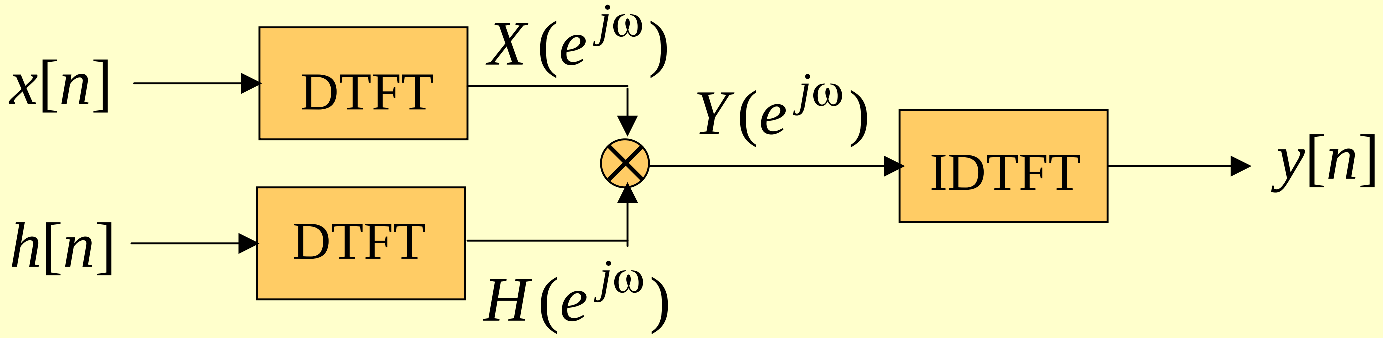
- An important property of the DTFT is given by the convolution theorem in Table 3.2
- It states that if $y[n] = x[n] \circledast h[n]$, then the DTFT $Y(e^{j\omega})$ of $y[n]$ is given by

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- An implication of this result is that the linear convolution $y[n]$ of the sequences $x[n]$ and $h[n]$ can be performed as follows:

Linear Convolution Using DTFT

- 1) Compute the DTFTs $X(e^{j\omega})$ and $H(e^{j\omega})$ of the sequences $x[n]$ and $h[n]$, respectively
- 2) Form the DTFT $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$
- 3) Compute the IDFT $y[n]$ of $Y(e^{j\omega})$



Discrete Fourier Transform

- Definition - The simplest relation between a length-N sequence $x[n]$, defined for $0 \le n \le N-1$, and its DTFT $X(e^{j\omega})$ is obtained by uniformly sampling $X(e^{j\omega})$ on the ω-axis between $0 \le \omega \le 2\pi$ at $\omega_k = 2\pi k / N$, $0 \le k \le N-1$

- From the definition of the DTFT we thus have

$$X[k] = X(e^{j\omega})\Big|_{\omega = 2\pi k/N} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi k/N},$$

$$0 \le k \le N-1$$

Discrete Fourier Transform

- <u>Note</u>: $X[k]$ is also a length-N sequence in the frequency domain
- The sequence $X[k]$ is called the **discrete Fourier transform** (DFT) of the sequence $x[n]$
- Using the notation $W_N = e^{-j2\pi/N}$ the DFT is usually expressed as:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \ 0 \le k \le N-1$$

Discrete Fourier Transform

- The **inverse discrete Fourier transform** (IDFT) is given by

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn}, \quad 0 \le n \le N-1$$

- To verify the above expression we multiply both sides of the above equation by $W_N^{\ell n}$ and sum the result from $n = 0$ to $n = N-1$

Discrete Fourier Transform

resulting in

$$\sum_{n=0}^{N-1} x[n]W_N^{\ell n} = \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}\right)W_N^{\ell n}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} X[k]W_N^{-(k-\ell)n}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=0}^{N-1} X[k]W_N^{-(k-\ell)n}$$

Discrete Fourier Transform

- Making use of the identity

$$\sum_{n=0}^{N-1} W_N^{-(k-\ell)n} = \begin{cases} N, \text{ for } k-\ell = rN, & r \text{ an integer} \\ 0, & \text{otherwise} \end{cases}$$

  we observe that the RHS of the last equation is equal to $X[\ell]$

- Hence

$$\sum_{n=0}^{N-1} x[n] W_N^{\ell n} = X[\ell]$$

Discrete Fourier Transform

- Example - Consider the length-N sequence

$$x[n] = \begin{cases} 1, & n = 0 \\ 0, & 1 \le n \le N-1 \end{cases}$$

- Its N-point DFT is given by

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = x[0] W_N^0 = 1$$

$$0 \le k \le N-1$$

Discrete Fourier Transform

- Example - Consider the length-N sequence

$$y[n] = \begin{cases} 1, & n = m \\ 0, & 0 \le n \le m-1,\ m+1 \le n \le N-1 \end{cases}$$

- Its N-point DFT is given by

$$Y[k] = \sum_{n=0}^{N-1} y[n] W_N^{kn} = y[m] W_N^{km} = W_N^{km}$$

$$0 \le k \le N-1$$

Discrete Fourier Transform

- Example - Consider the length-N sequence defined for $0 \leq n \leq N-1$

$$g[n] = \cos(2\pi rn/N), \;\; 0 \leq r \leq N-1$$

- Using a trigonometric identity we can write

$$g[n] = \frac{1}{2}\left(e^{j2\pi rn/N} + e^{-j2\pi rn/N}\right)$$

$$= \frac{1}{2}\left(W_N^{-rN} + W_N^{rN}\right)$$

Discrete Fourier Transform

- The N-point DFT of $g[n]$ is thus given by

$$G[k] = \sum_{n=0}^{N-1} g[n] W_N^{kn}$$

$$= \frac{1}{2}\left(\sum_{n=0}^{N-1} W_N^{-(r-k)n} + \sum_{n=0}^{N-1} W_N^{(r+k)n} \right),$$

$$0 \le k \le N-1$$

Discrete Fourier Transform

- Making use of the identity

$$\sum_{n=0}^{N-1} W_N^{-(k-\ell)n} = \begin{cases} N, \text{ for } k-\ell = rN, & r \text{ an integer} \\ 0, & \text{otherwise} \end{cases}$$

we get

$$G[k] = \begin{cases} N/2, & \text{for } k = r \\ N/2, & \text{for } k = N-r \\ 0, & \text{otherwise} \end{cases}$$

$$0 \le k \le N-1$$

Matrix Relations

- The DFT samples defined by

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le N-1$$

can be expressed in matrix form as

$$\mathbf{X} = \mathbf{D}_N \mathbf{x}$$

where

$$\mathbf{X} = \left[X[0] \quad X[1] \quad \cdots\cdots \quad X[N-1] \right]^T$$

$$\mathbf{x} = \left[x[0] \quad x[1] \quad \cdots\cdots \quad x[N-1] \right]^T$$

Matrix Relations

and $\mathbf{D}_N$ is the $N \times N$ **DFT matrix** given by

$$\mathbf{D}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}$$

Matrix Relations

- Likewise, the IDFT relation given by

$$x[n] = \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \; 0 \le n \le N-1$$

can be expressed in matrix form as

$$\mathbf{x} = \mathbf{D}_N^{-1} \mathbf{X}$$

where $\mathbf{D}_N^{-1}$ is the $N \times N$ **IDFT matrix**

Matrix Relations

where

$$\mathbf{D}_N^{-1} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix}$$

- Note:

$$\mathbf{D}_N^{-1} = \frac{1}{N}\mathbf{D}_N^{*}$$

DFT Computation Using MATLAB

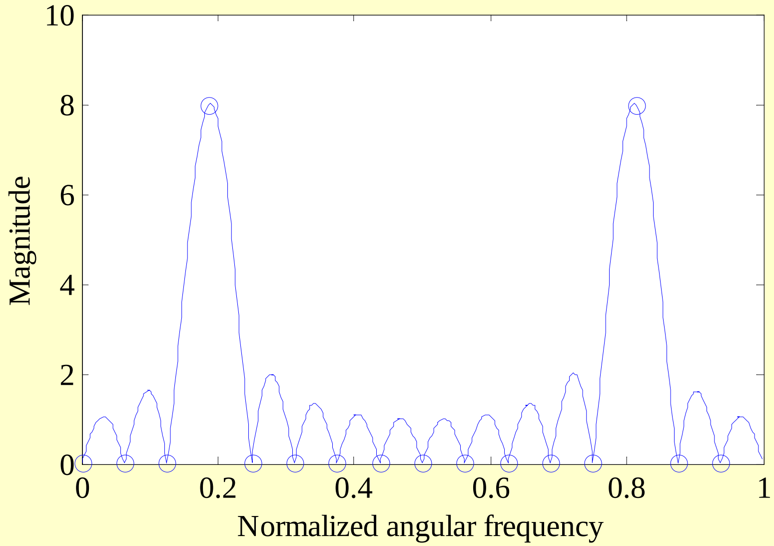
- The functions to compute the DFT and the IDFT are `fft` and `ifft`
- These functions make use of FFT algorithms which are computationally highly efficient compared to the direct computation
- Programs 3_2 and 3_4 illustrate the use of these functions

DFT Computation Using MATLAB

- <u>Example</u> - Program 3_4 can be used to compute the DFT and the DTFT of the sequence

$$x[n] = \cos(6\pi n/16), \quad 0 \le n \le 15$$

as shown below



○ indicates DFT samples

DTFT from DFT by Interpolation

- The N-point DFT $X[k]$ of a length-N sequence $x[n]$ is simply the frequency samples of its DTFT $X(e^{j\omega})$ evaluated at N uniformly spaced frequency points

$$\omega = \omega_k = 2\pi k / N, \quad 0 \le k \le N-1$$

- Given the N-point DFT $X[k]$ of a length-N sequence $x[n]$, its DTFT $X(e^{j\omega})$ can be uniquely determined from $X[k]$

DTFT from DFT by Interpolation

- Thus

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \right] e^{-j\omega n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \underbrace{\sum_{n=0}^{N-1} e^{-j(\omega - 2\pi k/N)n}}_{S}$$

DTFT from DFT by Interpolation

- To develop a compact expression for the sum S, let

$$r = e^{-j(\omega - 2\pi k/N)}$$

$$= \sum_{n=1}^{N-1} r^n + r^N - 1 = S + r^N - 1$$

- Then $\mathrm{S} = \sum_{n=0}^{N-1} r^n$
- From the above

$$r\mathrm{S} = \sum_{n=1}^{N} r^n = 1 + \sum_{n=1}^{N-1} r^n + r^N - 1$$

$$= \sum_{n=1}^{N-1} r^n + r^N - 1 = S + r^N - 1$$

DTFT from DFT by Interpolation

- Or, equivalently,

$$\mathrm{S} - r\mathrm{S} = (1-r)\mathrm{S} = 1 - r^N$$

- Hence

$$\mathrm{S} = \frac{1-r^N}{1-r} = \frac{1-e^{-j(\omega N-2\pi k)}}{1-e^{-j[\omega-(2\pi k/N)]}}$$

$$= \frac{\sin\left(\dfrac{\omega N-2\pi k}{2}\right)}{\sin\left(\dfrac{\omega N-2\pi k}{2N}\right)} \cdot e^{-j[(\omega-2\pi k/N)][(N-1)/2]}$$

DTFT from DFT by Interpolation

- Therefore

$$X(e^{j\omega})$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k] \frac{\sin\left(\frac{\omega N - 2\pi k}{2}\right)}{\sin\left(\frac{\omega N - 2\pi k}{2N}\right)} \cdot e^{-j[(\omega - 2\pi k/N)][(N-1)/2]}$$

Sampling the DTFT

- Consider a sequence $x[n]$ with a DTFT $X(e^{j\omega})$
- We sample $X(e^{j\omega})$ at N equally spaced points $\omega_k = 2\pi k / N$, $0 \le k \le N-1$ developing the N frequency samples $\{X(e^{j\omega_k})\}$
- These N frequency samples can be considered as an N-point DFT $Y[k]$ whose N-point IDFT is a length-N sequence $y[n]$

Sampling the DTFT

- Now $$X(e^{j\omega}) = \sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j\omega\ell}$$
- Thus $$Y[k] = X(e^{j\omega_k}) = X(e^{j2\pi k/N})$$
$$= \sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j2\pi k\ell/N} = \sum_{\ell=-\infty}^{\infty} x[\ell] W_N^{k\ell}$$
- An IDFT of $Y[k]$ yields
$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} Y[k] W_N^{-kn}$$

Sampling the DTFT

- i.e. $$y[n] = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{\ell=-\infty}^{\infty} x[\ell] W_N^{k\ell} W_N^{-kn}$$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell]\left[\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-\ell)}\right]$$

- Making use of the identity

$$\frac{1}{N}\sum_{n=0}^{N-1} W_N^{-k(n-r)} = \begin{cases} 1, & \text{for } r = n + mN \\ 0, & \text{otherwise} \end{cases}$$

Sampling the DTFT

we arrive at the desired relation

$$y[n] = \sum_{m=-\infty}^{\infty} x[n+mN], \quad 0 \le n \le N-1$$

- Thus $y[n]$ is obtained from $x[n]$ by adding an infinite number of shifted replicas of $x[n]$, with each replica shifted by an integer multiple of N sampling instants, and observing the sum only for the interval $0 \le n \le N-1$

Sampling the DTFT

- To apply

$$y[n] = \sum_{m=-\infty}^{\infty} x[n+mN], \;\; 0 \le n \le N-1$$

to finite-length sequences, we assume that the samples outside the specified range are zeros

- Thus if $x[n]$ is a length-M sequence with $M \le N$, then $y[n] = x[n]$ for $0 \le n \le N-1$

Sampling the DTFT

- If $M > N$, there is a time-domain aliasing of samples of $x[n]$ in generating $y[n]$, and $x[n]$ cannot be recovered from $y[n]$
- Example - Let $\{x[n]\} = \{\underset{\uparrow}{0} \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\}$
- By sampling its DTFT $X(e^{j\omega})$ at $\omega_k = 2\pi k/4$, $0 \le k \le 3$ and then applying a 4-point IDFT to these samples, we arrive at the sequence $y[n]$ given by

Sampling the DTFT

$$y[n] = x[n] + x[n+4] + x[n-4]\ ,\ 0 \le n \le 3$$

- i.e.

$$\{y[n]\} = \{\underset{\uparrow}{4} \quad 6 \quad 2 \quad 3\}$$

$\Rightarrow$ $\{x[n]\}$ cannot be recovered from $\{y[n]\}$

Numerical Computation of the DTFT Using the DFT

- A practical approach to the numerical computation of the DTFT of a finite-length sequence
- Let $X(e^{j\omega})$ be the DTFT of a length-N sequence $x[n]$
- We wish to evaluate $X(e^{j\omega})$ at a dense grid of frequencies $\omega_k = 2\pi k / M$, $0 \le k \le M - 1$, where $M >> N$:

Numerical Computation of the DTFT Using the DFT

$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n} = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/M}$$

- Define a new sequence

$$x_e[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ 0, & N \le n \le M-1 \end{cases}$$

- Then

$$X(e^{j\omega_k}) = \sum_{n=0}^{M-1} x[n]e^{-j2\pi kn/M}$$

Numerical Computation of the DTFT Using the DFT

- Thus $X(e^{j\omega_k})$ is essentially an M-point DFT $X_e[k]$ of the length-M sequence $x_e[n]$
- The DFT $X_e[k]$ can be computed very efficiently using the FFT algorithm if M is an integer power of 2
- The function `freqz` employs this approach to evaluate the frequency response at a prescribed set of frequencies of a DTFT expressed as a rational function in $e^{-j\omega}$

DFT Properties

- Like the DTFT, the DFT also satisfies a number of properties that are useful in signal processing applications
- Some of these properties are essentially identical to those of the DTFT, while some others are somewhat different
- A summary of the DFT properties are given in tables in the following slides

Table 3.5: General Properties of DFT

Type of Property	Length-N Sequence	N-point DFT
	$g[n]$ $h[n]$	$G[k]$ $H[k]$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G[k] + \beta H[k]$
Circular time-shifting	$g[\langle n - n_o \rangle_N]$	$W_N^{kn_o} G[k]$
Circular frequency-shifting	$W_N^{-k_o n} g[n]$	$G[\langle k - k_o \rangle_N]$
Duality	$G[n]$	$N g[\langle -k \rangle_N]$
N-point circular convolution	$\sum_{m=0}^{N-1} g[m] h[\langle n - m \rangle_N]$	$G[k]H[k]$
Modulation	$g[n]h[n]$	$\frac{1}{N} \sum_{m=0}^{N-1} G[m] H[\langle k - m \rangle_N]$
Parseval's relation	$\sum_{n=0}^{N-1} \lvert x[n] \rvert^2 = \frac{1}{N} \sum_{k=0}^{N-1} \lvert X[k] \rvert^2$	

Table 3.6: DFT Properties: Symmetry Relations

Length-N Sequence	N-point DFT
$x[n]$	$X[k]$
$x^*[n]$	$X^*[\langle -k\rangle_N]$
$x^*[\langle -n\rangle_N]$	$X^*[k]$
$\mathrm{Re}\{x[n]\}$	$X_{\mathrm{pcs}}[k] = \frac{1}{2}\{X[\langle k\rangle_N] + X^*[\langle -k\rangle_N]\}$
$j\,\mathrm{Im}\{x[n]\}$	$X_{\mathrm{pca}}[k] = \frac{1}{2}\{X[\langle k\rangle_N] - X^*[\langle -k\rangle_N]\}$
$x_{\mathrm{pcs}}[n]$	$\mathrm{Re}\{X[k]\}$
$x_{\mathrm{pca}}[n]$	$j\,\mathrm{Im}\{X[k]\}$

Note: $x_{\mathrm{pcs}}[n]$ and $x_{\mathrm{pca}}[n]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $x[n]$, respectively. Likewise, $X_{\mathrm{pcs}}[k]$ and $X_{\mathrm{pca}}[k]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $X[k]$, respectively.

$x[n]$ is a complex sequence

Table 3.7: DFT Properties: Symmetry Relations

Length-N Sequence	N-point DFT
$x[n]$	$X[k] = \text{Re}\{X[k]\} + j\,\text{Im}\{X[k]\}$
$x_{pe}[n]$	$\text{Re}\{X[k]\}$
$x_{po}[n]$	$j\,\text{Im}\{X[k]\}$
Symmetry relations	$X[k] = X^*[\langle -k\rangle_N]$ $\text{Re}\,X[k] = \text{Re}\,X[\langle -k\rangle_N]$ $\text{Im}\,X[k] = -\text{Im}\,X[\langle -k\rangle_N]$ $\lvert X[k]\rvert = \lvert X[\langle -k\rangle_N]\rvert$ $\arg X[k] = -\arg X[\langle -k\rangle_N]$

Note: $x_{pe}[n]$ and $x_{po}[n]$ are the periodic even and periodic odd parts of $x[n]$, respectively.

$x[n]$ is a real sequence

Circular Shift of a Sequence

- This property is analogous to the time-shifting property of the DTFT as given in Table 3.2, but with a subtle difference
- Consider length-N sequences defined for

$$0 \le n \le N-1$$

- Sample values of such sequences are equal to zero for values of $n < 0$ and $n \ge N$

Circular Shift of a Sequence

- If $x[n]$ is such a sequence, then for any arbitrary integer n_o, the shifted sequence

$$x_1[n] = x[n - n_o]$$

  is no longer defined for the range $0 \leq n \leq N - 1$
- We thus need to define another type of a shift that will always keep the shifted sequence in the range $0 \leq n \leq N - 1$

Circular Shift of a Sequence

- The desired shift, called the **circular shift**, is defined using a modulo operation:

$$x_c[n] = x[\langle n - n_o \rangle_N]$$

- For $n_o > 0$ (right circular shift), the above equation implies

$$x_c[n] = \begin{cases} x[n - n_o], & \text{for } n_o \le n \le N-1 \\ x[N - n_o + n], & \text{for } 0 \le n < n_o \end{cases}$$

Circular Shift of a Sequence

- Illustration of the concept of a circular shift

$x[n]$ $\quad$ $x[\langle n-1\rangle_6]$ $\quad$ $x[\langle n-4\rangle_6]$

$= x[\langle n+5\rangle_6]$ $\quad$ $= x[\langle n+2\rangle_6]$

Circular Shift of a Sequence

- As can be seen from the previous figure, a right circular shift by n_o is equivalent to a left circular shift by $N - n_o$ sample periods
- A circular shift by an integer number n_o greater than N is equivalent to a circular shift by $\langle n_o \rangle_N$

Circular Convolution

- This operation is analogous to linear convolution, but with a subtle difference
- Consider two length-N sequences, $g[n]$ and $h[n]$, respectively
- Their linear convolution results in a length-$(2N-1)$ sequence $y_L[n]$ given by

$$y_L[n] = \sum_{m=0}^{N-1} g[m]h[n-m], \quad 0 \le n \le 2N-2$$

Circular Convolution

- In computing $y_L[n]$ we have assumed that both length-N sequences have been zero-padded to extend their lengths to $2N-1$
- The longer form of $y_L[n]$ results from the time-reversal of the sequence $h[n]$ and its linear shift to the right
- The first nonzero value of $y_L[n]$ is $y_L[0] = g[0]h[0]$, and the last nonzero value is $y_L[2N-2] = g[N-1]h[N-1]$

Circular Convolution

- To develop a convolution-like operation resulting in a length-N sequence $y_C[n]$, we need to define a circular time-reversal, and then apply a circular time-shift
- Resulting operation, called a **circular convolution**, is defined by

$$y_C[n] = \sum_{m=0}^{N-1} g[m]h[\langle n-m \rangle_N], \quad 0 \le n \le N-1$$

Circular Convolution

- Since the operation defined involves two length-N sequences, it is often referred to as an N-point circular convolution, denoted as

$$y[n] = g[n] \, \textcircled{\scriptsize N} \, h[n]$$

- The circular convolution is commutative, i.e.

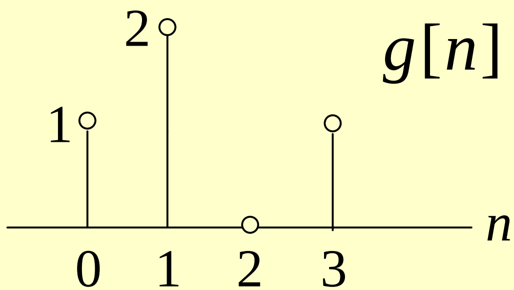
$$g[n] \, \textcircled{\scriptsize N} \, h[n] = h[n] \, \textcircled{\scriptsize N} \, g[n]$$

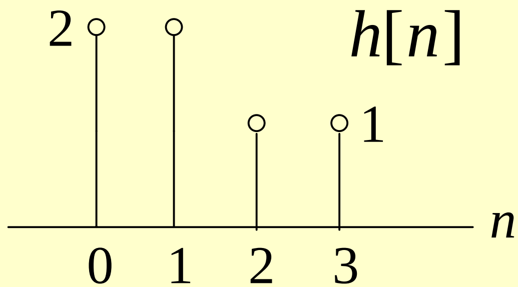
Circular Convolution

- <u>Example</u> - Determine the 4-point circular convolution of the two length-4 sequences:

$$\{g[n]\} = \{\underset{\uparrow}{1} \quad 2 \quad 0 \quad 1\}, \quad \{h[n]\} = \{\underset{\uparrow}{2} \quad 2 \quad 1 \quad 1\}$$

as sketched below





Circular Convolution

- The result is a length-4 sequence $y_C[n]$ given by

$$y_C[n] = g[n] \circledast_4 h[n] = \sum_{m=0}^{3} g[m]h[\langle n-m \rangle_4],$$

$$0 \le n \le 3$$

- From the above we observe

$$y_C[0] = \sum_{m=0}^{3} g[m]h[\langle -m \rangle_4]$$

$$= g[0]h[0] + g[1]h[3] + g[2]h[2] + g[3]h[1]$$

$$= (1\times 2) + (2\times 1) + (0\times 1) + (1\times 2) = 6$$

Circular Convolution

- Likewise $y_C[1] = \sum_{m=0}^{3} g[m]h[\langle 1-m\rangle_4]$

$$= g[0]h[1] + g[1]h[0] + g[2]h[3] + g[3]h[2]$$

$$= (1\times 2) + (2\times 2) + (0\times 1) + (1\times 1) = 7$$

$$y_C[2] = \sum_{m=0}^{3} g[m]h[\langle 2-m\rangle_4]$$

$$= g[0]h[2] + g[1]h[1] + g[2]h[0] + g[3]h[3]$$

$$= (1\times 1) + (2\times 2) + (0\times 2) + (1\times 1) = 6$$

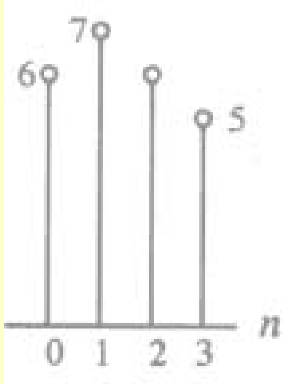
Circular Convolution

and

$$y_C[3] = \sum_{m=0}^{3} g[m]h[\langle 3-m\rangle_4]$$

$$= g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0]$$

$$= (1\times 1) + (2\times 1) + (0\times 2) + (1\times 2) = 5$$

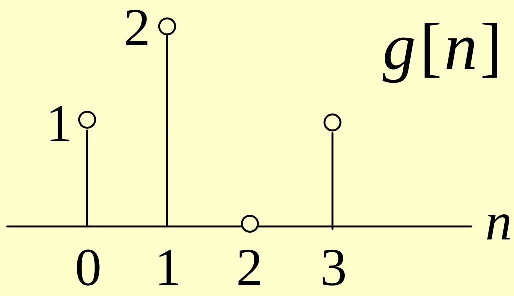


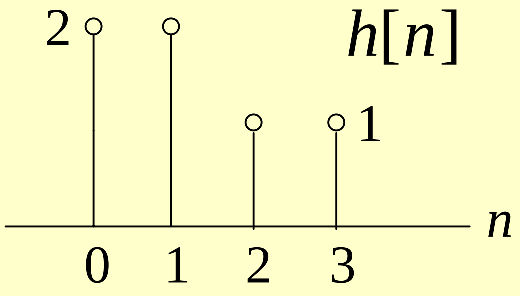
$y_C[n]$

- The circular convolution can also be computed using a DFT-based approach as indicated in Table 3.5

Circular Convolution

- <u>Example</u> - Consider the two length-4 sequences repeated below for convenience:





- The 4-point DFT $G[k]$ of $g[n]$ is given by

$$G[k] = g[0] + g[1]e^{-j2\pi k/4} + g[2]e^{-j4\pi k/4} + g[3]e^{-j6\pi k/4}$$

$$= 1 + 2e^{-j\pi k/2} + e^{-j3\pi k/2}, \quad 0 \le k \le 3$$

Circular Convolution

- Therefore $G[0] = 1 + 2 + 1 = 4,$

$$G[1] = 1 - j2 + j = 1 - j,$$

$$G[2] = 1 - 2 - 1 = -2,$$

$$G[3] = 1 + j2 - j = 1 + j$$

- Likewise,

$$H[k] = h[0] + h[1]e^{-j2\pi k/4}$$

$$+ h[2]e^{-j4\pi k/4} + h[3]e^{-j6\pi k/4}$$

$$= 2 + 2e^{-j\pi k/2} + e^{-j\pi k} + e^{-j3\pi k/2}, \quad 0 \le k \le 3$$

Circular Convolution

- Hence, $H[0] = 2 + 2 + 1 + 1 = 6,$

$$H[1] = 2 - j2 - 1 + j = 1 - j,$$

$$H[2] = 2 - 2 + 1 - 1 = 0,$$

$$H[3] = 2 + j2 - 1 - j = 1 + j$$

- The two 4-point DFTs can also be computed using the matrix relation given earlier

Circular Convolution

$$\begin{bmatrix} G[0] \\ G[1] \\ G[2] \\ G[3] \end{bmatrix} = \mathbf{D}_4 \begin{bmatrix} g[0] \\ g[1] \\ g[2] \\ g[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix}$$

$$\begin{bmatrix} H[0] \\ H[1] \\ H[2] \\ H[3] \end{bmatrix} = \mathbf{D}_4 \begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

$\mathbf{D}_4$ is the 4-point DFT matrix

Circular Convolution

- If $Y_C[k]$ denotes the 4-point DFT of $y_C[n]$ then from Table 3.5 we observe

$$Y_C[k] = G[k]H[k], \quad 0 \le k \le 3$$

- Thus

$$\begin{bmatrix} Y_C[0] \\ Y_C[1] \\ Y_C[2] \\ Y_C[3] \end{bmatrix} = \begin{bmatrix} G[0]H[0] \\ G[1]H[1] \\ G[2]H[2] \\ G[3]H[3] \end{bmatrix} = \begin{bmatrix} 24 \\ -j2 \\ 0 \\ j2 \end{bmatrix}$$

Circular Convolution

- A 4-point IDFT of $Y_C[k]$ yields

$$\begin{bmatrix} y_C[0] \\ y_C[1] \\ y_C[2] \\ y_C[3] \end{bmatrix} = \frac{1}{4}\mathbf{D}_4^* \begin{bmatrix} Y_C[0] \\ Y_C[1] \\ Y_C[2] \\ Y_C[3] \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 24 \\ -j2 \\ 0 \\ j2 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 6 \\ 5 \end{bmatrix}$$

Circular Convolution

- Example - Now let us extended the two length-4 sequences to length 7 by appending each with three zero-valued samples, i.e.

$$g_e[n] = \begin{cases} g[n], & 0 \le n \le 3 \\ 0, & 4 \le n \le 6 \end{cases}$$

$$h_e[n] = \begin{cases} h[n], & 0 \le n \le 3 \\ 0, & 4 \le n \le 6 \end{cases}$$

Circular Convolution

- We next determine the 7-point circular convolution of $g_e[n]$ and $h_e[n]$:

$$y[n] = \sum_{m=0}^{6} g_e[m] h_e[\langle n-m \rangle_7], \quad 0 \le n \le 6$$

- From the above $y[0] = g_e[0]h_e[0] + g_e[1]h_e[6]$

$$+ g_e[3]h_e[4] + g_e[4]h_e[3] + g_e[5]h_e[2] + g_e[6]h_e[1]$$

$$= g[0]h[0] = 1 \times 2 = 2$$

Circular Convolution

- Continuing the process we arrive at

$$y[1] = g[0]h[1] + g[1]h[0] = (1\times 2) + (2\times 2) = 6,$$

$$y[2] = g[0]h[2] + g[1]h[1] + g[2]h[0]$$

$$= (1\times 1) + (2\times 2) + (0\times 2) = 5,$$

$$y[3] = g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0]$$

$$= (1\times 1) + (2\times 1) + (0\times 2) + (1\times 2) = 5,$$

$$y[4] = g[1]h[3] + g[2]h[2] + g[3]h[1]$$

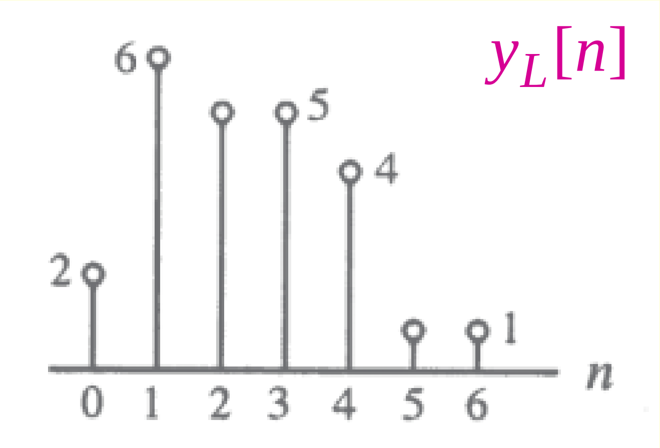
$$= (2\times 1) + (0\times 1) + (1\times 2) = 4,$$

Circular Convolution

$$y[5] = g[2]h[3] + g[3]h[2] = (0\times 1) + (1\times 1) = 1,$$

$$y[6] = g[3]h[3] = (1\times 1) = 1$$

- As can be seen from the above that $y[n]$ is precisely the sequence $y_L[n]$ obtained by a linear convolution of $g[n]$ and $h[n]$



Circular Convolution

- The N-point circular convolution can be written in matrix form as

$$\begin{bmatrix} y_C[0] \\ y_C[1] \\ y_C[2] \\ \vdots \\ y_C[N-1] \end{bmatrix} = \begin{bmatrix} h[0] & h[N-1] & h[N-2] & \cdots & h[1] \\ h[1] & h[0] & h[N-1] & \cdots & h[2] \\ h[2] & h[1] & h[0] & \cdots & h[3] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h[N-1] & h[N-2] & h[N-3] & \cdots & h[0] \end{bmatrix} \begin{bmatrix} g[0] \\ g[1] \\ g[2] \\ \vdots \\ g[N-1] \end{bmatrix}$$

- Note: The elements of each diagonal of the $N \times N$ matrix are equal
- Such a matrix is called a **circulant matrix**